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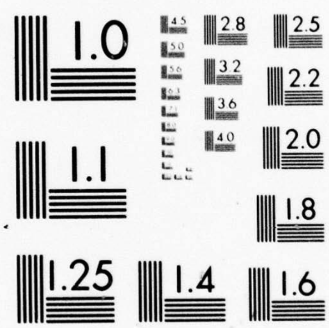

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DISCONTINUOUS DEFORMATION GRADIENTS IN PLANE FINITE ELASTOSTATICS OF INCOMPRESSIBLE MATERIALS

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Discontinuous deformation gradients in plane finite elastostatics of incompressible materials*

by

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Summary

This investigation is concerned with the possibility of the change of type of the differential equations governing finite plane elastostatics for incompressible elastic materials, and the related issue of the existence of equilibrium fields with discontinuous deformation gradients. Explicit necessary and sufficient conditions on the deformation invariants and the material for the ellipticity of the plane displacement equations of equilibrium are established. The issue of the existence, locally, of "elastostatic shocks" — elastostatic fields with continuous displacements and discontinuous deformation gradients — is then investigated. It is shown that an elastostatic shock exists only if the governing field equations suffer a loss of ellipticity at some deformation. Conversely, if the governing field equations have lost ellipticity at a given deformation at some point, an elastostatic shock can exist, locally, at that point. The results obtained are valid for an arbitrary homogeneous, isotropic, incompressible, elastic material.

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1.1 Introduction

In two recent papers [1], [2], Knowles and Sternberg looked into the question of the possible loss of ellipticity of the displacement equations of equilibrium of nonlinear elastostatics for compressible materials. In [1], three dimensional homogeneous deformations of a particular isotropic compressible elastic material were considered, and necessary and sufficient restrictions on the principal stretches for ellipticity to prevail were deduced. It was shown that for this material, a loss of ellipticity occurred at sufficiently severe local deformations. In [2] they established similar explicit necessary and sufficient conditions for an arbitrary homogeneous, isotropic, compressible elastic solid subjected to plane deformations.

These papers were motivated by some asymptotic studies of crack problems they had considered previously, in which certain difficulties encountered suggested that the problem may not admit a classically smooth solution.

In a subsequent paper [3] Knowles and Sternberg investigated the implications of a loss of ellipticity. The question of the existence of "elastostatic (or equilibrium) shocks" — solutions which possess finite jump discontinuities of the displacement gradient across certain surfaces while maintaining continuous displacements — was studied within the context of plane deformations of compressible elastic solids. It was established in [3] that a necessary condition for the existence of a piecewise homogeneous elastostatic shock was that the material lose

strong ellipticity at some homogeneous deformation. The question of whether in fact a loss of ordinary ellipticity was necessary was left unanswered. In the particular case of weak elastostatic shocks it was shown that ordinary ellipticity must necessarily be lost at the preassigned deformation on one side of the shock.

Rice [5] had previously examined the phenomenon of "localization of deformation" for plastic materials. Localization is the bifurcation of an initially smooth state of deformation into one involving a zone of highly localized shearing. Localized deformations as described in [5] appear to have certain qualitative features in common with elastostatic shocks as described in [3]. In fact, within his setting, Rice shows that the onset of localization is first possible, in a program of deformation, when the displacement equations of equilibrium lose ellipticity.

In the present study we treat the corresponding issues for an arbitrary homogeneous incompressible elastic solid subjected to plane deformations. Some of the results established are appropriate only for isotropic materials. Explicit necessary and sufficient restrictions on the deformation invariants and the material are deduced which ensure ellipticity of the plane displacement equations of equilibrium. In the context of isotropic materials it is established that a loss of ordinary ellipticity at some homogeneous deformation is a necessary condition for the existence of a piecewise homogeneous elastostatic shock. It is further shown that a strict loss of ordinary ellipticity at a given homogeneous deformation is sufficient, but not necessary, to ensure the existence of a piecewise homogeneous elastostatic shock which has associated with it this preassigned deformation on one side.

In Section 2 we cite some relevant results from the theory of finite elastostatics for incompressible elastic solids which we then specialize to plane deformations. The notion of the "local amount of shear" associated with any plane volume preserving deformation is then described. In Section 3 the conventional notion of ellipticity is adapted to the displacement equations of equilibrium and necessary and sufficient conditions for ellipticity are then deduced. In the isotropic case these conditions are put into explicit form and a simple interpretation is given in terms of what we call the "local amount of shear". A loss of ellipticity is found to depend on a loss of invertibility of the shear stress-amount of shear relation in simple shear.

The notion of piecewise homogeneous elastostatic shocks developed in [3] for the compressible case is extended to the incompressible case in Section 4. In Section 5 we then consider <u>weak</u> elastostatic shocks in homogeneous, incompressible, <u>anisotropic</u> elastic solids and show that a loss of ellipticity at the pre-assigned deformation on one side of the shock is necessary for its existence. The jumps across the shock-line of various physically significant field quantities are also deduced.

In Section 6 we return to equilibrium shocks of <u>finite</u> strength in homogeneous, incompressible, <u>isotropic</u> elastic solids. We show that a strict failure of ordinary ellipticity at a given deformation is sufficient to ensure the existence of a piecewise homogeneous elastostatic shock which has associated with it this deformation on one side. Moreover we show that a failure of ordinary ellipticity at some homogeneous deformation is necessary for the existence of a shock of the type under consideration.

In Section 7 we discuss the dissipativity inequality first proposed by Knowles and Sternberg in [3] and later extended by Knowles [4] to

three-dimensional deformations of both compressible and incompressible materials and explore some of its consequences. In particular its implications in the case of weak elastostatic shocks in anisotropic materials is examined.

Finally in Section 8 we illustrate some of the preceeding results by means of an example involving a particular hypothetical constitutive law.

2.1 Preliminaries on Finite Plane Elastostatics

Let $\mathcal{R}$ be the three-dimensional open region occupied by the interior of a body in its undeformed configuration. A deformation of the body is described by a sufficiently smooth and invertible transformation

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \quad \text{on } \mathcal{R} \tag{2.1}$$

which maps $\mathcal{R}$ onto a domain $\mathcal{R}_*$. Here $\underset{\sim}{y}$ is the position vector after deformation of the particle which, in the undeformed configuration was located at $\underset{\sim}{x}$. We will assume for the moment that the displacement vector field $\underset{\sim}{u}(\underset{\sim}{x})$ is twice continuously differentiable on $\mathcal{R}$.

The deformation gradient tensor $\underset{\sim}{F}$ is defined by

$$\underset{\sim}{F} = \nabla \underset{\sim}{y} \quad \text{on } \mathcal{R}, \tag{2.2}$$

and since the material is presumed to be incompressible,

$$\det \underset{\sim}{F} = 1 \quad \text{on } \mathcal{R}, \tag{2.3}$$

where $\det \underset{\sim}{F}$ is the Jacobian of the mapping (2.1). The right and left Cauchy-Green tensors $\underset{\sim}{C}$ and $\underset{\sim}{G}$ are defined respectively by

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F}, \quad \underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T. \tag{2.4}$$

Let $\underset{\sim}{\tau}$ be the Cauchy stress tensor field accompanying the deformation at hand. Assuming that $\underset{\sim}{\tau}$ is continuously differentiable on $\mathcal{R}_*$, the equilibrium equations are

$$\operatorname{div} \underset{\sim}{\tau} = \underset{\sim}{0} \ , \quad \underset{\sim}{\tau} = \underset{\sim}{\tau}^T \quad \text{on } \mathcal{R}_* \ , \tag{2.5}$$

where body forces are presumed to be absent. The nominal (Piola) stress tensor corresponding to $\underset{\sim}{\tau}$ is given by

$$\underset{\sim}{\sigma} = \underset{\sim}{\tau}(\underset{\sim}{F}^T)^{-1} \ , \tag{2.6}$$

where use has been made of (2.3). Equations (2.2), (2.3), (2.5) and (2.6) lead to the equilibrium equations

$$\operatorname{div} \underset{\sim}{\sigma} = \underset{\sim}{0} \ , \quad \underset{\sim}{\sigma}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{\sigma}^T \quad \text{on } \mathcal{R} \ . \tag{2.7}$$

We now turn to the constitutive relations and suppose that the body is homogeneous and elastic and possesses an elastic potential $W = \hat{W}(\underset{\sim}{F})$. W represents the strain energy density per unit undeformed volume. The nominal stresses are now given by

$$\underset{\sim}{\sigma} = \hat{W}_{\underset{\sim}{F}}(\underset{\sim}{F}) - p(\underset{\sim}{F}^T)^{-1} \ , \tag{2.8}$$ [1]

where $p(\underset{\sim}{x})$ is a scalar field arising because of the incompressibility constraint. We assume for the moment that $p(\underset{\sim}{x})$ is continuously differentiable on $\mathcal{R}$. Alternatively, from (2.6), (2.8) we have

$$\underset{\sim}{\tau} = \hat{W}_{\underset{\sim}{F}}(\underset{\sim}{F})\underset{\sim}{F}^T - p\underset{\sim}{1} \ . \tag{2.9}$$

From (2.1)-(2.3), (2.7) and (2.8) we are led to

[1] See Truesdell and Noll [6], page 304.

$$c_{ijk\ell}(\underset{\sim}{F})u_{k,\ell j} - p_{,j}F^{-1}_{ji} = 0 \quad \text{on } \mathcal{R} \,, \tag{2.10}$$[1]

where

$$c_{ijk\ell}(\underset{\sim}{F}) = \frac{\partial^2 \hat{W}(\underset{\sim}{F})}{\partial F_{ij}\partial F_{k\ell}} \,. \tag{2.11}$$

Let $\lambda_1^2(\underset{\sim}{x})$, $\lambda_2^2(\underset{\sim}{x})$ and $\lambda_3^2(\underset{\sim}{x})$, where $\lambda_i > 0$, be the eigenvalues of the symmetric positive definite tensor field $\underset{\sim}{G}$ (or $\underset{\sim}{C}$). The principal scalar invariants of $\underset{\sim}{G}$ are

$$I_1 = \operatorname{tr}\underset{\sim}{G} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \,, \quad I_2 = \tfrac{1}{2}[(\operatorname{tr}\underset{\sim}{G})^2 - (\operatorname{tr}\underset{\sim}{G}^2)] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \,,$$

$$I_3 = \det\underset{\sim}{G} = \lambda_1^2\lambda_2^2\lambda_3^2 \,, \tag{2.12}$$

where tr denotes the trace. From (2.3), (2.4) and (2.12) it follows that

$$\lambda_1^2\lambda_2^2\lambda_3^2 = 1 \quad \text{on } \mathcal{R} \,. \tag{2.13}$$

In the special case when the material is incompressible and isotropic, W depends on $\underset{\sim}{F}$ only through the invariants I_1 and I_2, whence we have

---

[1] All tensor and vector components are taken with respect to a fixed rectangular cartesian frame. A comma followed by a subscript indicates differentiation with respect to the corresponding x-coordinate. Latin subscripts take the values 1, 2, 3 while Greek subscripts take the values 1, 2. Repeated subscripts are summed over the appropriate range.

$$W = \overset{*}{W}(I_1, I_2) \quad . \tag{2.14}$$

Suppose now that the domain $\mathcal{R}$ occupied by the undeformed body is a right cylinder with generators parallel to the x_3-axis. Let Π be the open region of the (x_1, x_2)-plane occupied by the interior of the middle cross-section of this cylinder. Suppose further that the deformation (2.1) is a plane deformation so that

$$y_\alpha = x_\alpha + u_\alpha(x_1, x_2) \ , \ y_3 = x_3 \quad \text{on } \mathcal{R} \quad . \tag{2.15}$$

Π is then mapped onto a domain Π_* of the same plane, which would be the middle cross-section of the cylindrical region $\mathcal{R}_*$. From here on we shall be exclusively concerned with plane deformations unless specifically stated otherwise. It follows from (2.2) and (2.15) that

$$F_{\alpha\beta} = y_{\alpha,\beta} \ , \ F_{\alpha 3} = F_{3\alpha} = 0 \ , \ F_{33} = 1 \quad . \tag{2.16}$$

The nominal stresses are now given by

$$\sigma_{\alpha\beta} = \frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}} - pF^{-1}_{\beta\alpha} \ , \ \sigma_{33} = \frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{33}} - p \quad . \tag{2.17}$$

If we assume that the elastic potential W is such that

$$\frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha 3}} = \frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{3\alpha}} = 0 \tag{2.18}$$

for every $\underset{\sim}{F}$ such that (2.16) holds, then we further have

$$\sigma_{3\alpha} = \sigma_{\alpha 3} = 0 \ . \tag{2.19}$$

The assumption (2.18) holds true identically for isotropic materials in particular.

One sees readily from (2.7), (2.15)-(2.19) that for equilibrium in the x_3-direction it is necessary and sufficient that the scalar field $p(\underset{\sim}{x})$ be independent of x_3. Thus

$$p = p(x_1, x_2) \quad \text{on } \Pi \ . \tag{2.20}$$

In the present circumstances (2.10) specializes to

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) u_{\gamma,\beta\delta} - p_{,\beta} F^{-1}_{\beta\alpha} = 0 \quad \text{on } \Pi \ . \tag{2.21}$$

Equation (2.21), together with the incompressibility condition (2.3), constitute the governing system of equations for the plane strain problem and we shall refer to them as the <u>displacement equations of equilibrium</u> in plane strain. They are three scalar equations involving the three functions $u_\alpha(x_1, x_2)$ and $p(x_1, x_2)$.

One sees readily from (2.4) and (2.15) that in any plane deformation, unity is an eigenvalue of the left and right Cauchy-Green tensors, whence we have

$$\lambda_3(\underset{\sim}{x}) = 1 \ . \tag{2.22}$$

Equations (2.12) and (2.13) now specialize to

$$I_1 = \lambda_1^2 + \lambda_2^2 + 1 \ , \ I_2 = \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2 \ , \ I_3 = \lambda_1^2\lambda_2^2 \ , \tag{2.23}$$

and

$$\lambda_1^2\lambda_2^2 = 1 \ , \tag{2.24}$$

whence

$$I_1 = I_2 = \lambda_1^2 + \frac{1}{\lambda_1^2} + 1 \ . \tag{2.25}$$

If we now define I by

$$I = F_{\alpha\beta}F_{\alpha\beta} \ , \tag{2.26}$$

we have, because of (2.4), (2.12), (2.16) and (2.25) that

$$I = I_1 - 1 = I_2 - 1 = 2 + \left(\lambda_1 - \frac{1}{\lambda_1}\right)^2 \geq 2 \ . \tag{2.27}$$

In the special case when the material is isotropic, we have from (2.14) and (2.27) that, in plane deformations,

$$W = \overset{*}{W}(I+1 \ , \ I+1) \tag{2.29}$$

so that if we define the <u>Plane Strain Elastic Potential</u> $W(I)$ by

$$W(I) = \overset{*}{W}(I+1\ ,\ I+1)\ ,\qquad I \geq 2\ , \tag{2.30}$$

we have in the present context that $\hat{W}(\underset{\sim}{F}) = W(I)$ where $I = F_{\alpha\beta}F_{\alpha\beta}$. It follows from this that

$$\frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}} = 2F_{\alpha\beta}W'(I)\ , \tag{2.31}$$

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) = \frac{\partial^2 \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}} = 2\delta_{\alpha\gamma}\delta_{\beta\delta}W'(I) + 4F_{\alpha\beta}F_{\gamma\delta}W''(I)\ . \tag{2.32}$$

From (2.4), (2.9) and (2.31) we conclude that

$$\tau_{\alpha\beta} = 2W'(I)G_{\alpha\beta} - p\delta_{\alpha\beta}\ . \tag{2.33}$$

It is apparent that the plane strain elastic potential $W(I)$ fully determines the in-plane stress components. This is not true, however, of the component τ_{33} .

Finally we recall that in this case the in-plane Baker-Ericksen inequality requires that

$$(\tau_1 - \tau_2)(\lambda_1 - \lambda_2) > 0 \quad \text{if } \lambda_1 \neq \lambda_2 \tag{2.34}$$ [1]

for all pure homogeneous (plane) deformations of the form

$$y_\alpha = \lambda_\alpha x_\alpha \quad \text{(no sum)};\ \lambda_1\lambda_2 = 1\ ,\ \lambda_\alpha > 0\ ,\ y_3 = x_3\ , \tag{2.35}$$

[1] See Truesdell and Noll [6], page 158.

where τ_α are the principal in-plane Cauchy stresses. From (2.4), (2.16), (2.26), (2.33) and (2.35) we have

$$\tau_\alpha = 2W'(I)\lambda_\alpha^2 - p \ , \quad I = \lambda_1^2 + \lambda_2^2 \ , \tag{2.36}$$

whence (2.34) may be equivalently written as

$$W'(\lambda_1^2+\lambda_2^2)(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)>0 \ , \quad \lambda_1,\lambda_2>0 \ , \ \lambda_1\lambda_2=1, \ \lambda_1\neq\lambda_2 \tag{2.37}$$

or

$$W'(\lambda_1^2+\lambda_2^2)>0 \ , \quad \lambda_1,\lambda_2>0 \ , \ \lambda_1\lambda_2=1 \ , \ \lambda_1\neq\lambda_2 \ , \tag{2.38}$$

which in turn is equivalent to

$$W'(I)>0 \ \text{ for } \ I>2 \ . \tag{2.39}$$

The infinitesimal shear modulus is easily shown to be $\mathring{\mu} = 2W'(2)$; if we assume that $\mathring{\mu}>0$, we may replace (2.39) by

$$W'(I)>0 \ \text{ for } \ I\geq 2 \ . \tag{2.40}$$

Requiring that (2.40) hold for the material at hand is equivalent to requiring that the material have a positive (finite) shear modulus. Conversely, (2.40) implies (2.34), though it does not imply the full (three-dimensional) Baker-Ericksen inequalities.

2.2 Local Amount of Shear

We now establish that <u>any plane volume preserving deformation can be decomposed locally into the product of a simple shear in a suitable direction followed or preceded by a suitable rotation</u>.

To this end, let $\underset{\sim}{F}$ be a two-dimensional tensor such that $\det \underset{\sim}{F} = 1$. Define

$$k = \sqrt{I - 2} \ , \quad I = F_{\alpha\beta}F_{\alpha\beta} \ . \qquad (2.41)^1$$

Then we will show that there exist proper orthogonal tensors $\underset{\sim}{Q}_1$, $\underset{\sim}{Q}_2$, non-singular tensors $\underset{\sim}{K}_1$, $\underset{\sim}{K}_2$ with unit determinant (all two-dimensional) and rectangular cartesian frames X_1, X_2 such that

$$\underset{\sim}{F} = \underset{\sim}{Q}_1\underset{\sim}{K}_1 = \underset{\sim}{K}_2\underset{\sim}{Q}_2 \ , \qquad (2.42)$$

$$\underset{\sim}{K}_1^{X_1} = \underset{\sim}{K}_2^{X_2} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \ . \qquad (2.43)^2$$

Conversely, if (2.42) holds for some proper orthogonal tensors $\underset{\sim}{Q}_1$, $\underset{\sim}{Q}_2$ and tensors $\underset{\sim}{K}_1$, $\underset{\sim}{K}_2$ with unit determinant such that (2.43) is true in some rectangular cartesian frames X_1, X_2, then we will show that k is necessarily given by $k = \pm\sqrt{I - 2}$.

In order to prove the first part of the result, let X be a principal frame for the symmetric positive definite tensor $\underset{\sim}{F}\underset{\sim}{F}^T$. Then

$$(\underset{\sim}{F}\underset{\sim}{F}^T)^X = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{bmatrix} \ , \quad \lambda > 0 \qquad (2.44)$$

---

[1] Since $\det \underset{\sim}{F} = 1$, we have that necessarily $I \geq 2$.

[2] $\underset{\sim}{K}_1^{X_1}$ is the matrix of components of the tensor $\underset{\sim}{K}_1$ in the frame $\underset{\sim}{X}_1$.

where we have made use of the fact that $\det \underset{\sim}{F} = 1$. Clearly we may assume $\lambda \geq 1$ with no loss of generality. Consider the rectangular cartesian coordinate frame X_2 obtained by rotating the frame X counterclockwise through an angle θ determined by

$$\sin\theta = -\frac{1}{\sqrt{1+\lambda^2}}\ , \quad \cos\theta = \frac{\lambda}{\sqrt{1+\lambda^2}}\ . \tag{2.45}$$

By the change of frame formula for tensors,

$$(\underset{\sim}{F}\underset{\sim}{F}^T)^{X_2} = \underset{\sim}{R}(\underset{\sim}{F}\underset{\sim}{F}^T)^X \underset{\sim}{R}^T\ , \quad \underset{\sim}{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

we compute $(\underset{\sim}{F}\underset{\sim}{F}^T)^{X_2}$ to find

$$(\underset{\sim}{F}\underset{\sim}{F}^T)^{X_2} = \begin{bmatrix} \lambda^2+\lambda^{-2}-1 & \lambda-\lambda^{-1} \\ \lambda-\lambda^{-1} & 1 \end{bmatrix}\ . \tag{2.46}$$

Let $\underset{\sim}{K}_2$ be the tensor with unit determinant defined by

$$\underset{\sim}{K}_2^{X_2} = \begin{bmatrix} 1 & \lambda-\lambda^{-1} \\ 0 & 1 \end{bmatrix}\ . \tag{2.47}$$

Then (2.46) and (2.47) imply that

$$\underset{\sim}{F}\underset{\sim}{F}^T = \underset{\sim}{K}_2\underset{\sim}{K}_2^T \ . \tag{2.48}$$

Define the tensor $\underset{\sim}{Q}_2$ by

$$\underset{\sim}{Q}_2 = \underset{\sim}{K}_2^{-1}\underset{\sim}{F} \ ; \tag{2.49}$$

(2.48) and (2.49) now lead to

$$\underset{\sim}{K}_2\underset{\sim}{Q}_2\underset{\sim}{Q}_2^T\underset{\sim}{K}_2^T = \underset{\sim}{K}_2\underset{\sim}{K}_2^T \ .$$

Since $\underset{\sim}{K}_2$ is non-singular it thus follows that $\underset{\sim}{Q}_2\underset{\sim}{Q}_2^T = \underset{\sim}{1}$ whence $\underset{\sim}{Q}_2$ is orthogonal. But, from (2.49) it follows that $\det \underset{\sim}{Q}_2 = +1$ since $\det \underset{\sim}{K}_2 = \det \underset{\sim}{F} = 1$, so that in fact $\underset{\sim}{Q}_2$ is proper orthogonal.

Finally, since we are assuming $\lambda \geq 1$, it follows from (2.46) that $\lambda - \lambda^{-1} = \sqrt{F_{\alpha\beta}F_{\alpha\beta} - 2} = \sqrt{I - 2}$ whence from (2.41) $k = \lambda - \lambda^{-1}$. This establishes the left decomposition $\underset{\sim}{F} = \underset{\sim}{K}_2\underset{\sim}{Q}_2$. The right decomposition $\underset{\sim}{F} = \underset{\sim}{Q}_1\underset{\sim}{K}_1$ can be similarly established by considering $\underset{\sim}{F}^T\underset{\sim}{F}$ in place of $\underset{\sim}{F}\underset{\sim}{F}^T$.

The second part of the result is easily proved as follows. Suppose now that associated with the given tensor $\underset{\sim}{F}$ there exists some proper orthogonal tensor $\underset{\sim}{Q}_2$, some tensor $\underset{\sim}{K}_2$ with unit determinant and some rectangular cartesian frame X_2 such that

$$\underset{\sim}{F} = \underset{\sim}{K}_2\underset{\sim}{Q}_2 \ , \tag{2.50}$$

$$\underset{\approx}{K}_2^{X_2} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} , \tag{2.51}$$

for some real number k. Note that the tensors $\underset{\approx}{Q}_2$, $\underset{\approx}{K}_2$ and the frame X_2 are not required to be the particular ones used in the preceeding proof. Since $\underset{\approx}{Q}_2$ is orthogonal, it follows from (2.50) that

$$\underset{\approx}{F}\underset{\approx}{F}^T = \underset{\approx}{K}_2\underset{\approx}{K}_2^T , \tag{2.52}$$

whence in particular, the traces of the two-dimensional tensors $\underset{\approx}{F}\underset{\approx}{F}^T$ and $\underset{\approx}{K}_2\underset{\approx}{K}_2^T$ are equal. By virtue of (2.51) we now have that necessarily $I = F_{\alpha\beta}F_{\alpha\beta} = 2 + k^2$, whence

$$k = \pm\sqrt{I - 2} .$$

The corresponding result for the decomposition $\underset{\sim}{F} = \underset{\approx}{Q}_1\underset{\approx}{K}_1$ may be similarly established.

Given any plane volume preserving deformation with deformation gradient $\underset{\sim}{F}(\underset{\sim}{x})$, we refer to $k(\underset{\sim}{x})$ defined by (2.41) as the associated <u>local amount of shear</u>. Therefore any <u>arbitrary</u> plane deformation of an incompressible material can be viewed <u>locally</u> as a simple shear in a suitable direction with local amount of shear $k(\underset{\sim}{x})$, followed or preceded by a suitable rotation.

3.1 Ellipticity of the Plane Displacement Equations of Equilibrium

We now introduce the relevant notion of ellipticity without restricting ourselves to isotropic materials.

Consider a cylindrical surface S with generators parallel to those of the undeformed body and lying wholly within $\mathcal{R}$. Let C be the curve along which S intersects Π . Assume that C has a continuous curvature, and let ξ be the arc length on C . Then C may be described by the non-singular parameterization

$$C:\ x_\alpha = \mathring{x}_\alpha(\xi) \ .$$

If ζ is a coordinate normal to C and $\underset{\sim}{N}(\xi)$ is a unit vector normal to C in the (x_1, x_2)-plane, then near a fixed point P on C we have the orthogonal curvilinear coordinate system (ξ, ζ), permitting us to write

$$x_\alpha = \mathring{x}_\alpha(\xi) + \zeta N_\alpha(\xi) \tag{3.1}$$

for any point (x_1, x_2) in a two-dimensional neighborhood of P. The mapping (3.1) is locally one to one, so that it has an inverse

$$\xi = f(x_1, x_2) \ , \quad \zeta = g(x_1, x_2) \ , \tag{3.2}$$

and f and g are twice continuously differentiable in a neighborhood of P . Note that we may take

$$\underset{\sim}{N} = \frac{\nabla \underset{\sim}{g}}{|\nabla \underset{\sim}{g}|} \quad \text{on } C \ .$$

Now suppose that $(u_\alpha(x_1, x_2)\ ,\ p(x_1, x_2))$ is a solution of the plane displacement equations of equilibrium (2.3) and (2.21) such that u_α is once continuously differentiable and twice piecewise continuously differentiable on Π, while p is continuous and piecewise continuously differentiable on Π. We set

$$\hat{u}_\alpha(\xi, \zeta) = u_\alpha(\mathring{x}_1(\xi) + \zeta N_1(\xi)\ ,\ \mathring{x}_2(\xi) + \zeta N_2(\xi))\ ,$$

$$\hat{p}(\xi, \zeta) = p(\mathring{x}_1(\xi) + \zeta N_1(\xi)\ ,\ \mathring{x}_2(\xi) + \zeta N_2(\xi))\ ,$$

and further suppose that, in fact, the second order partial derivatives of $\hat{u}_\alpha$ are all continuous across C except possibly for the normal derivative $\partial^2\hat{u}_\alpha/\partial\zeta^2$, and that the first order partial derivative $\partial\hat{p}/\partial\xi$ is continuous across C, while the normal derivative $\partial\hat{p}/\partial\zeta$ may suffer a jump discontinuity.

Let

$$U_\alpha = \left[\frac{\partial^2\hat{u}_\alpha}{\partial\zeta^2}\right] \quad , \quad q = \left[\frac{\partial\hat{p}}{\partial\zeta}\right] \tag{3.3}$$

where $[h]$ denotes the jump of a function h across C. Then one shows easily that

$$[u_{\alpha,\beta\gamma}] = U_\alpha \bar{N}_\beta \bar{N}_\gamma \ , \qquad (3.4)^1$$

where $\bar{\underset{\sim}{N}} = \nabla \underset{\sim}{g} = (\bar{\underset{\sim}{N}} \cdot \bar{\underset{\sim}{N}})^{1/2} \underset{\sim}{N}$. We have by the chain rule and (3.2) that

[1]See Section 1 of [1].

$$p_{,\alpha} = \frac{\partial\hat{p}}{\partial\xi}\frac{\partial f}{\partial x_\alpha} + \frac{\partial\hat{p}}{\partial\zeta}\frac{\partial g}{\partial x_\alpha} \ ,$$

which because of the presumed smoothness and (3.3) leads to

$$[p_{,\alpha}] = q\bar{N}_\alpha \ . \tag{3.5}$$

Taking jumps in the first two displacement equations of equilibrium (2.21), and making use of (3.4), (3.5) and the assumed smoothness we get

$$c_{\alpha\beta\gamma\delta}U_\gamma\bar{N}_\delta\bar{N}_\beta - F^{-1}_{\beta\alpha}\bar{N}_\beta q = 0 \quad \text{on } C \ . \tag{3.6}$$

If for all vectors $\underset{\sim}{\bar{N}}$ and nonsingular tensors $\underset{\sim}{F}$ with unit determinant, we define the matrix $Q_{\alpha\beta}(\underset{\sim}{\bar{N}}, \underset{\sim}{F})$ through

$$Q_{\alpha\gamma}(\underset{\sim}{\bar{N}}, \underset{\sim}{F}) = c_{\alpha\beta\gamma\delta}(\underset{\sim}{F})\bar{N}_\beta\bar{N}_\delta \ , \tag{3.7}$$

then $Q_{\alpha\beta}$ is symmetric by virtue of (2.11). Equation (3.6) can now be written in the form

$$Q_{\alpha\beta}U_\beta = qF^{-1}_{\beta\alpha}\bar{N}_\beta \quad \text{on } C \ . \tag{3.8}$$

We also need the "jump equation" associated with the remaining displacement equation of equilibrium (2.3). We compute $\partial(\det\underset{\sim}{F})/\partial\zeta$ to find

$$\frac{\partial}{\partial\zeta}(\det\underset{\sim}{F}) = (\det\underset{\sim}{F})F^{-1}_{\beta\alpha}\frac{\partial}{\partial\zeta}\left\{\frac{\partial\hat{u}_\alpha}{\partial\xi}\frac{\partial f}{\partial x_\beta} + \frac{\partial\hat{u}_\alpha}{\partial\zeta}\frac{\partial g}{\partial x_\beta}\right\} \ , \tag{3.9}$$

where use has been made of (3.2), the chain rule and a standard formula

for the differentiation of a determinant. Taking jumps in (3.9) and making use of (2.3), (3.3) and the presumed smoothness leads to

$$\left[\frac{\partial}{\partial \zeta}(\det \underset{\sim}{F})\right] = F^{-1}_{\beta\alpha}\bar{N}_\beta U_\alpha \ . \tag{3.10}$$

But by (2.3) the jump in $\det \underset{\sim}{F}$ must vanish, whence (3.10) simplifies to

$$F^{-1}_{\beta\alpha}\bar{N}_\beta U_\alpha = 0 \quad \text{on } C \ . \tag{3.11}$$

The system of jump equations associated with the displacement equations of equilibrium are (3.8) and (3.11), and may be regarded as three linear homogeneous algebraic equations for the jumps U_α and q.

We say that the system of plane displacement equations of equilibrium is <u>elliptic</u> at the solution (u_α, p) and at the point (x_1, x_2) if and only if, for all vectors $\underset{\sim}{\bar{N}} \neq \underset{\sim}{0}$, the system (3.8), (3.11) has only the trivial solution $U_\alpha = 0$, $q = 0$.

Consequently if the system is elliptic, the displacement field u_α will in fact be twice continuously differentiable at the point under consideration and the pressure p will be continuously differentiable there. If on the other hand there exists a non-trivial solution of (3.8), (3.11) for some vector $\underset{\sim}{\bar{N}}$, then $\underset{\sim}{\bar{N}}$ is normal to a characteristic curve in the undeformed configuration. These characteristic curves are the only possible carriers of discontinuities of the kind admitted here in u_α and p, and ellipticity precludes the existence of real characteristics.

If we set

$$m_\alpha = F^{-1}_{\beta\alpha}\bar{N}_\beta \ , \tag{3.12}$$

we can write the system of jump equations (3.8) and (3.11) as

$$\begin{bmatrix} Q_{11} & Q_{12} & -m_1 \\ Q_{21} & Q_{22} & -m_2 \\ m_1 & m_2 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ q \end{bmatrix} = \underset{\sim}{0} \ .$$

This system of linear homogeneous algebraic equations for U_α and q has only the trivial solution if and only if

$$\det \begin{bmatrix} Q_{11} & Q_{12} & -m_1 \\ Q_{21} & Q_{22} & -m_2 \\ m_1 & m_2 & 0 \end{bmatrix} \neq 0 \ , \tag{3.13}$$

or equivalently

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}Q_{\alpha\beta}m_\lambda m_\mu \neq 0 \ . \qquad (3.14)^1$$

Since $\underset{\sim}{F}$ has unit determinant, one shows easily that in plane strain

$$F^{-1}_{\beta\alpha} = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}F_{\gamma\delta} \ . \tag{3.15}$$

By virtue of (3.12) and (3.15) we may write (3.14) equivalently as

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}F_{\gamma\lambda}F_{\delta\mu}Q_{\gamma\delta}\bar{N}_\alpha\bar{N}_\beta \neq 0 \ . \tag{3.16}$$

Therefore, we have that <u>a necessary and sufficient condition for</u>

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[1] $\epsilon_{\alpha\beta}$ is the two-dimensional alternator. $\epsilon_{11}=\epsilon_{22}=0$, $\epsilon_{12}=1$, $\epsilon_{21}=-1$.

<u>the displacement equations of equilibrium to be elliptic</u> at a solution (u_α, p) and at some point (x_1, x_2) is

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}F_{\gamma\lambda}F_{\delta\mu}Q_{\gamma\delta}(\underset{\sim}{\overline{N}}, \underset{\sim}{F})\overline{N}_\alpha\overline{N}_\beta \neq 0 \ , \tag{3.17}$$

for <u>every</u> vector $\underset{\sim}{\overline{N}} \neq \underset{\sim}{0}$. Finally, because of (3.7) it is clear that (3.17) is equivalent to

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}F_{\gamma\lambda}F_{\delta\mu}Q_{\gamma\delta}(\underset{\sim}{N}, \underset{\sim}{F})N_\alpha N_\beta \neq 0 \qquad \text{for all } \underline{\text{unit}} \text{ vectors } \underset{\sim}{N} \ . \tag{3.18}$$

3.2 <u>Specialization to Isotropic Materials</u>

When the material at hand is isotropic, we can use (2.4), (2.11), (2.32) and (3.7) to simplify the necessary and sufficient condition for ellipticity (3.18), which then gives

$$(\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}C_{\alpha\beta}N_\lambda N_\mu)W'(I) + 2(\epsilon_{\alpha\lambda}C_{\alpha\rho}N_\rho N_\lambda)^2 W''(I) \neq 0 \ , \tag{3.19}$$

for every unit vector $\underset{\sim}{N}$. Now let the frame be principal for $\underset{\sim}{C}$, so that

$$[C_{\alpha\beta}] = \begin{bmatrix} \lambda_1^2 & 0 \\ & \\ 0 & \lambda_2^2 \end{bmatrix} ,$$

and evaluate (3.19) in this frame. We then find

$$(\lambda_1^2 N_2^2 + \lambda_2^2 N_1^2)W'(I) + 2(\lambda_1^2 - \lambda_2^2)^2 N_1^2 N_2^2 W''(I) \neq 0 \tag{3.20}$$

for all unit vectors $\underset{\sim}{N}$, as being necessary and sufficient for ellipticity.

We will now show that <u>the plane displacement equations of</u>

equilibrium are elliptic at a solution (u_α, p) and at a point (x_1, x_2) if and only if

$$W'(I) \neq 0 \ , \ \frac{2W''(I)}{W'(I)}(I-2)+1>0 \ , \qquad (3.21)$$

at the point under consideration; i.e. that (3.21) is equivalent to (3.20).

To show this, we observe that since $\underset{\sim}{N}$ is a unit vector,

$$\lambda_1^2 N_2^2 + \lambda_2^2 N_1^2 = (\lambda_1^2 N_2^2 + \lambda_2^2 N_1^2)(N_1^2 + N_2^2) = \lambda_1^2 N_2^4 + \lambda_2^2 N_1^4 + (\lambda_1^2 + \lambda_2^2) N_1^2 N_2^2 \ , \qquad (3.22)$$

so that (3.20) may be written as

$$\{\lambda_2^2 W'(I)\} N_1^4 + \{\lambda_1^2 W'(I)\} N_2^4 + \{(\lambda_1^2 + \lambda_2^2) W'(I) + 2(\lambda_1^2 - \lambda_2^2)^2 W''(I)\} N_1^2 N_2^2 \neq 0 \qquad (3.23)$$

for all unit vectors $\underset{\sim}{N}$. If we set

$$\begin{gathered} E_{11} = \lambda_2^2 W'(I) \ , \ E_{22} = \lambda_1^2 W'(I) \ , \\ E_{21} = E_{12} = \frac{(\lambda_1^2 + \lambda_2^2)}{2} W'(I) + (\lambda_1^2 - \lambda_2^2)^2 W''(I) \ , \ z_\alpha = N_\alpha^2 \ , \end{gathered} \qquad (3.24)$$

we can replace (3.23) by

$$E_{\alpha\beta} z_\alpha z_\beta \neq 0 \quad \text{for all } \underset{\sim}{z} \neq \underset{\sim}{0} \ , \ z_\alpha \geq 0 \ . \qquad (3.25)$$

It has been shown in Section 2 of reference [2] that (3.25) holds if and only if

$$E_{11}E_{22} > 0 \tag{3.26}$$

and

$$\frac{\epsilon\, E_{12}}{\sqrt{E_{11}E_{22}}} > -1 \ , \tag{3.27}$$

where

$$\epsilon = \operatorname{sgn}\ E_{11} = \operatorname{sgn}\ E_{22} \ . \tag{3.28}$$

Substituting from (3.24) into (3.26) we get $\lambda_1^2\lambda_2^2\{W'(I)\}^2 > 0$ which, because $\lambda_\alpha > 0$, is equivalent to

$$W'(I) \neq 0 \ . \tag{3.29}$$

Using (3.24) and (3.28) in (3.27) leads to

$$2(\lambda_1 - \lambda_2)^2 \frac{W''(I)}{W'(I)} + 1 > 0 \ ,$$

which because of (2.24) and (2.27) may in turn be written as

$$2(I-2)\frac{W''(I)}{W'(I)} + 1 > 0 \ . \tag{3.30}$$

Equations (3.29) and (3.30) are what we set out to establish.

A physical interpretation of the ellipticity condition (3.21) may be obtained in terms of the concept of the local amount of shear introduced in Section 2.2. Consider an isotropic, incompressible, homogeneous, elastic solid which has a positive shear modulus:[1]

[1]See (2.40). A similar interpretation can clearly be given in terms of the local amount of shear even in the unrealistic case when (2.40) does not hold.

$$W'(I)>0 \quad \text{for } I \geq 2 \ . \tag{3.31}$$

The first of (3.21) is now trivially satisfied. If we define the function τ by

$$\tau(k) = 2kW'(2+k^2) \ , \quad |k|<\infty \ , \tag{3.32}$$

then $\tau(k)$ is easily shown to be the shear stress corresponding to an amount of shear k in a simple shear deformation. The graph of $\tau(k)$ vs. k described by (3.32) will be called the <u>response curve in simple shear</u>. Differentiating (3.32) with respect to k and observing that (3.31) holds leads to

$$\tau'(k) = 2W'(2+k^2)\left\{2k^2 \frac{W''(2+k^2)}{W'(2+k^2)} + 1\right\} \ .$$

We therefore find that (3.21) is equivalent to

$$\tau'(k)>0 \quad \text{for } k=\sqrt{I-2} \ , \tag{3.33}$$

from which we conclude that <u>for an isotropic, incompressible elastic solid having a positive shear modulus, the plane displacement equations of equilibrium are elliptic at a solution</u> (u_α, p) <u>and a point</u> (x_1, x_2) <u>if and only if the slope of the response curve in simple shear at an amount of shear equal to the local amount of shear is positive</u>.

Suppose for example that the response of a particular homogeneous, isotropic, incompressible elastic solid in simple shear is as described by Fig. 2. Then in <u>any</u> plane deformation the displacement equations of equilibrium are elliptic at some point (x_1, x_2) and some solution if and only if the local amount of shear at that point $k(x_1, x_2)$,

defined by (2.41), is such that $-k_0 < k(x_1, x_2) < k_0$.

It is apparent from the above discussion that a loss of ellipticity for materials of the type being considered is dependent upon a loss of invertibility of the shear stress – amount of shear relation in simple shear.

Finally, we note from (3.21) that the undeformed state is elliptic if and only if the infinitesimal shear modulus $\hat{\mu} = 2W'(2) \neq 0$. This is precisely the condition for ellipticity of the linearized displacement equations of equilibrium for a homogeneous, isotropic, incompressible, elastic material.

3.3 Characteristic Curves

If the ellipticity condition (3.21) is violated, it follows that there exists a unit vector $\underset{\sim}{N}$ such that equality holds in (3.20). $\underset{\sim}{N}$ will then be normal to a (material) characteristic, and we now determine the number of possible characteristics and their inclinations. To this end, let

$$N_1 = -\sin\theta \quad , \quad N_2 = \cos\theta \quad , \tag{3.34}$$

so that θ is the local inclination of the material characteristic to the λ_1-principal axis of $\underset{\sim}{C}$. Substituting this in (3.20), with equality holding now, we find

$$(\lambda_1^2 \cos^2\theta + \lambda_2^2 \sin^2\theta) W'(I) + 2(\lambda_1^2 - \lambda_2^2)^2 \sin^2\theta \cos^2\theta \, W''(I) = 0 \quad . \tag{3.35}$$

We seek solutions θ of this equation in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let us assume that the infinitesimal shear modulus of the material is positive:

$$\mathring{\mu} = 2W'(2) > 0 \quad . \tag{3.36}$$

We see immediately from (3.21) that, if the point under consideration is locally undeformed $(I = 2)$ in the given deformation, then the displacement equations of equilibrium are elliptic there. Consequently we need only consider $I > 2$ in our search for characteristics.

Suppose first that ellipticity is lost by virtue of the fact that the first of the ellipticity conditions (3.21) is violated.[1] Then

$$W'(I) = 0 \tag{3.37}$$

at the point (x_1, x_2) of interest at the given deformation. We then find from (3.35) that either $W''(I) = 0$ or $\theta = 0, \frac{\pi}{2}$. Using (2.41) and (3.32), we may state this result as follows. Let k be the local amount of shear. Then if $\tau(k) = 0$, the displacement equations of equilibrium are not elliptic for the given deformation at the point under consideration. Furthermore, we then have two (material) characteristics inclined at angles 0 and $\frac{\pi}{2}$ to a principal axis of $\underset{\sim}{C}$, except in the particular case when $\tau'(k) = 0$ as well, in which case any number of arbitrarily inclined characteristics may exist locally.

Now suppose that $W'(I) \neq 0$ at the point of interest and that ellipticity is lost by virtue of the fact that the second of (3.21) has been violated. Then

$$\frac{2W''(I)}{W'(I)}(I - 2) + 1 \leq 0 \quad . \tag{3.38}$$

Equation (3.35) can now be rearranged into the form of a quadratic

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[1]Note from (2.40) that this possibility does not exist if the material has a positive shear modulus.

equation for $\cos 2\theta$.

$$\frac{(\lambda_1^2-\lambda_2^2)^2}{2}\frac{W''(I)}{W'(I)}\cos^2 2\theta - \frac{(\lambda_1^2-\lambda_2^2)}{2}\cos 2\theta$$

$$-\left\{\frac{\lambda_1^2+\lambda_2^2}{2}+\frac{(\lambda_1^2-\lambda_2^2)^2}{2}\frac{W''(I)}{W'(I)}\right\}=0 \quad . \qquad (3.39)$$

Formally we can write the solution of this after making use of (2.24) and (2.27) as

$$\cos 2\theta = \frac{1\pm(\{2(I-2)W''(I)/W'(I)+1\}\ \{2(I+2)W''(I)/W'(I)+1\})^{\frac{1}{2}}}{2(I^2-4)^{\frac{1}{2}}W''(I)/W'(I)} \qquad (3.40)$$

where with no loss of generality we have assumed that $\lambda_1 > \lambda_2$.

If (3.38) holds with equality (so that $\tau(k) \neq 0$, $\tau'(k)=0$ at the local amount of shear k) we find two values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ from (3.40), whence two characteristics exist. Equation (3.40) now simplifies to

$$\cos 2\theta = -\sqrt{\frac{I-2}{I+2}} \ , \qquad (3.41)$$

which because of (2.24) and (2.27) (and since $\lambda_1 > \lambda_2$) leads to

$$\cos 2\theta = \frac{1-\lambda_1^2}{1+\lambda_1^2} \ , \qquad (3.42)$$

whence

$$\tan\theta = \pm\lambda_1 \ . \qquad (3.43)$$

Suppose the <u>spatial</u> characteristic corresponding to this material characteristic is inclined at an angle α to the λ_1-principal axis of $\underset{\sim}{G}$. It can be shown that

$$\tan\alpha = \frac{\lambda_2}{\lambda_1}\tan\theta \quad , \tag{3.44}$$

so that (2.24), (3.43) and (3.44) give

$$\tan\alpha = \pm\frac{1}{\lambda_1} \quad . \tag{3.45}$$

Because of (2.33), α is also the inclination to the corresponding principal axis of the Cauchy stress tensor.

If however, strict inequality holds in (3.38) (so that $\tau(k) \neq 0$, $k\tau(k)\tau'(k) < 0$ at the local amount of shear k) (3.40) gives us four values of θ which in turn implies the existence of two pairs of characteristics. Clearly, each pair is positioned symmetrically with respect to the principal axes of $\underset{\sim}{G}$. In what follows we will have need for the inclinations α of the corresponding spatial characteristics to the λ_1-principal axis of $\underset{\sim}{G}$ $(\lambda_1 > \lambda_2)$. From (2.24), (2.27), (3.40) and (3.44) we have

$$\cos 2\alpha = \frac{-1 \pm (\{2(I-2)W''(I)/W'(I)+1\}\{2(I+2)W''(I)/W'(I)+1\})^{\frac{1}{2}}}{2(I^2-4)^{\frac{1}{2}}W''(I)/W'(I)} \quad . \tag{3.46}$$

4.1 Weak Formulation of Problem

In the derivation of the classical field equations of elasticity the displacement field $\underset{\sim}{u}$ and stress field $\underset{\sim}{\sigma}$ are assumed to satisfy certain smoothness requirements. There are, however, some physical problems in which these conditions are not met, so that in order to study them one would be forced to relax the smoothness demanded of the field quantities. It may, for example, be necessary to require only that the displacement field $\underset{\sim}{u}(\underset{\sim}{x})$ be continuous and piecewise continuously differentiable on $\mathcal{R}$, while the nominal stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ and the pressure field $p(\underset{\sim}{x})$ are to be piecewise continuous[1] on $\mathcal{R}$. Clearly, the global balance laws continue to be meaningful even under these smoothness conditions, but one must re-examine the validity of the local field equations.

Of particular physical interest is the case wherein the field quantities possess the classical degree of smoothness[2] everywhere except on one or more regular surfaces within the body. This would, for example, describe an idealized model for shear bands. To formulate this problem, we suppose that there is a surface S in $\mathcal{R}$ such that $\underset{\sim}{\sigma}$, $\underset{\sim}{F}$ and p are continuously differentiable everywhere in $\mathcal{R}$ except on S, and such that $\underset{\sim}{\sigma}$, $\underset{\sim}{F}$ and p suffer finite jump discontinuities across it. The displacement $\underset{\sim}{u}(\underset{\sim}{x})$ is presumed to be continuous

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[1] We return momentarily to the three-dimensional case in this section.

[2] See Section 2.1.

everywhere in $\mathcal{R}$. The possibility of the breakdown of ellipticity of the governing equations suggests that solutions of this type to the equations of finite elastostatics may emerge in some circumstances.

Going through the usual arguments,[1] one finds from the global equilibrium of forces that

$$\operatorname{div} \underset{\sim}{\sigma} = \underset{\sim}{0} \quad \text{on } \mathcal{R} - S \tag{4.1}$$

and

$$[\underset{\sim}{\sigma}]_{-}^{+} \underset{\sim}{N} = \underset{\sim}{0} \quad \text{on } S \ , \tag{4.2}$$

while from the global equilibrium of moments we have

$$\underset{\sim}{\sigma} \underset{\sim}{F}^{T} = \underset{\sim}{F} \underset{\sim}{\sigma}^{T} \quad \text{on } \mathcal{R} - S \tag{4.3}$$

and

$$\underset{\sim}{y}(\underset{\sim}{x}) \times [\underset{\sim}{\sigma}]_{-}^{+} \underset{\sim}{N} = \underset{\sim}{0} \quad \text{on } S \ . \tag{4.4}$$

Equation (4.2) says that the nominal tractions are continuous across S . Here $[\underset{\sim}{\sigma}]_{-}^{+} = \overset{+}{\underset{\sim}{\sigma}} - \overset{-}{\underset{\sim}{\sigma}}$ where $\overset{+}{\underset{\sim}{\sigma}}$ and $\overset{-}{\underset{\sim}{\sigma}}$ are the limiting values of $\underset{\sim}{\sigma}$ (presumed to exist) as a point on S is approached from each side, and $\underset{\sim}{N}$ is is a unit normal to S . Equations (4.2) and (4.4) are referred to as <u>jump conditions</u>. Note that (4.4) is trivially satisfied once (4.2) is. Incompressibility likewise leads to

$$\det \underset{\sim}{F} = 1 \quad \text{on } \mathcal{R} - S \ . \tag{4.5}$$

[1]See Chadwick [7], page 114.

Such a surface S carrying jump discontinuities in $\underset{\sim}{F}$, $\underset{\sim}{\sigma}$ and p which conform with the jump condition (4.2), while maintaining continuous displacements across it is called an "equilibrium shock", or an "elastostatic shock" in the particular case when the body is composed of an elastic material.

4.2 Piecewise Homogeneous Elastostatic Shocks

To investigate many of the local issues related to elastostatic shocks, it is sufficient to consider the case in which S is a plane and the deformation gradient $\underset{\sim}{F}$ is constant on either side of S. From here on we shall be concerned with such a situation within the context of plane deformations[1] of an incompressible elastic solid, so that we may take S to be a plane parallel to the generators of the body.

The corresponding problem for a compressible elastic solid was investigated by Knowles and Sternberg [3]. In this section, we formulate the problem governing the existence of an elastostatic shock in the incompressible case in a manner entirely analogous to [3].

Suppose that the middle cross-section of the body we are dealing with occupies the entire (x_1, x_2)-plane Π in its undeformed configuration. Let X be a fixed rectangular cartesian coordinate frame and let $\mathcal{L}$ be the straight line through the origin of X with unit direction vector $\underset{\sim}{L}$. Thus

$$\mathcal{L}: \; x_\alpha = L_\alpha \xi \; , \quad -\infty < \xi < \infty \; . \qquad (4.6)$$

[1] We leave the three-dimensional introduction to elastostatic shocks of the last section and return to plane deformations from here on.

Let $\underset{\sim}{N}$ be the unit vector normal to $\mathcal{L}$ obtained by a counterclockwise rotation of $\underset{\sim}{L}$. Let $\overset{+}{\Pi}$ and $\overset{-}{\Pi}$ be the two open half planes into which $\mathcal{L}$ divides Π, with $\overset{+}{\Pi}$ being the one into which $\underset{\sim}{N}$ points. (See Fig. 1.)

Now consider the piecewise homogeneous plane deformation

$$y_\alpha = \overset{+}{F}_{\alpha\beta} x_\beta \quad \text{on } \overset{+}{\Pi}, \quad y_\alpha = \overset{-}{F}_{\alpha\beta} x_\beta \quad \text{on } \overset{-}{\Pi}, \tag{4.7}$$

where $\underset{\sim}{\overset{+}{F}}$ and $\underset{\sim}{\overset{-}{F}}$ are constant tensors such that

$$\det \underset{\sim}{\overset{+}{F}} = \det \underset{\sim}{\overset{-}{F}} = 1 \quad . \tag{4.8}$$

The nominal stresses associated with the deformation (4.7) are

$$\overset{+}{\sigma}_{\alpha\beta} = \frac{\partial \hat{W}(\underset{\sim}{\overset{+}{F}})}{\partial F_{\alpha\beta}} - \overset{+}{p}\overset{+}{F}{}^{-1}_{\beta\alpha} \quad \text{on } \overset{+}{\Pi}, \quad \overset{-}{\sigma}_{\alpha\beta} = \frac{\partial \hat{W}(\underset{\sim}{\overset{-}{F}})}{\partial F_{\alpha\beta}} - \overset{-}{p}\overset{-}{F}{}^{-1}_{\beta\alpha} \quad \text{on } \overset{-}{\Pi} \quad . \tag{4.9}$$

Clearly, the equilibrium equations (4.1) are satisfied if and only if $\overset{+}{p}$ and $\overset{-}{p}$ are constants.

If we are to view the line $\mathcal{L}$ as the intersection of an equilibrium shock S with the cross-section Π, then according to Section 4.1 we need to impose displacement and traction continuity requirements across $\mathcal{L}$. Because of (4.7) the requirement of a continuous displacement field is equivalent to

$$\overset{+}{F}_{\alpha\beta} x_\beta = \overset{-}{F}_{\alpha\beta} x_\beta \quad \text{on } \mathcal{L}, \tag{4.10}$$

which in view of (4.6) reduces to

$$\overset{+}{F}_{\alpha\beta} L_\beta = \overset{-}{F}_{\alpha\beta} L_\beta \ . \tag{4.11}$$

By (4.9), we have traction continuity (4.2) if and only if

$$\left\{ \frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \overset{+}{p} \overset{+}{F}{}^{-1}_{\beta\alpha} \right\} N_\beta = \left\{ \frac{\partial \hat{W}(\overset{-}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \overset{-}{p} \overset{-}{F}{}^{-1}_{\beta\alpha} \right\} N_\beta \ . \tag{4.12}$$

If the deformation field (4.7), subject to (4.8), together with real constants $\overset{+}{p}$ and $\overset{-}{p}$ conform with (4.11) and (4.12), and if $\overset{+}{\underset{\sim}{F}} \neq \overset{-}{\underset{\sim}{F}}$, then we refer to the corresponding elastostatic field as a piecewise homogeneous elastostatic shock.[1] The line $\mathcal{L}$ will be referred to as the material shock-line. Figure 1(b) displays the images of the three rectangles shown in Fig. 1(a) under a typical mapping (4.7) in the presence of such a shock.

In order to examine questions related to the existence of piecewise homogeneous elastostatic shocks we pose the following problem. Given a constant tensor $\overset{+}{\underset{\sim}{F}}$ with $\det \overset{+}{\underset{\sim}{F}} = 1$ and a real constant $\overset{+}{p}$, determine a constant tensor $\overset{-}{\underset{\sim}{F}}$ with $\det \overset{-}{\underset{\sim}{F}} = 1$ $(\overset{-}{\underset{\sim}{F}} \neq \overset{+}{\underset{\sim}{F}})$ and a real constant $\overset{-}{p}$ such that (4.11) and (4.12) hold.

Equation (4.11) may be solved as follows. Let $\mathcal{L}_*$, which we shall refer to as the spatial shock-line, be the image of $\mathcal{L}$ under the mapping (4.7). Let $\overset{+}{\Pi}_*$ and $\overset{-}{\Pi}_*$ be the two half planes into which $\overset{+}{\Pi}$ and $\overset{-}{\Pi}$ map by virtue of (4.7). Suppose $\underset{\sim}{\ell}$ is the unit direction vector of $\mathcal{L}_*$ such that the unit normal $\underset{\sim}{n}$ to $\mathcal{L}_*$ obtained by rotating $\underset{\sim}{\ell}$ counterclockwise points into $\overset{+}{\Pi}_*$. (See Fig. 1.) Without any loss of

[1] Note from (4.12) that if $\overset{+}{p} \neq \overset{-}{p}$ then necessarily $\overset{+}{\underset{\sim}{F}} \neq \overset{-}{\underset{\sim}{F}}$.

generality the inclinations Φ and ϕ of the shock-lines $\mathcal{L}$ and $\mathcal{L}_*$ relative to the x_1-axis may be confined to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

One can show readily that, given a constant tensor $\overset{+}{\underset{\sim}{F}}$ with $\det \overset{+}{\underset{\sim}{F}} = 1$, (4.11) will hold for a tensor $\overset{-}{\underset{\sim}{F}}$ with unit determinant if and only if

$$\overset{-}{F}_{\alpha\beta} = (\delta_{\alpha\gamma} + \kappa \ell_\alpha n_\gamma)\overset{+}{F}_{\gamma\beta} , \tag{4.13}$$

for some real number κ. We omit the derivation of this result as it parallels exactly the corresponding derivation in the compressible case contained in [3]. Let X' be the rectangular cartesian frame obtained by rotating the frame X counterclockwise through an angle ϕ. The base vectors associated with X' are then $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$. Expressing (4.13) in the frame X' we have

$$\begin{bmatrix} \overset{-}{F}{}^{X'}_{11} & \overset{-}{F}{}^{X'}_{12} \\ \overset{-}{F}{}^{X'}_{21} & \overset{-}{F}{}^{X'}_{22} \end{bmatrix} = \begin{bmatrix} 1 & \kappa \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overset{+}{F}{}^{X'}_{11} & \overset{+}{F}{}^{X'}_{12} \\ \overset{+}{F}{}^{X'}_{21} & \overset{+}{F}{}^{X'}_{22} \end{bmatrix} . \tag{4.14}$$

Accordingly, the deformation on $\overset{-}{\Pi}$ may be viewed as being equivalent to the deformation on $\overset{+}{\Pi}$ followed by a simple shear parallel to $\mathcal{L}_*$ with an amount of shear κ.

We may now pose the following problem which is equivalent to the one posed earlier. Given a constant tensor $\overset{+}{\underset{\sim}{F}}$ with unit determinant and a real constant $\overset{+}{p}$, determine real numbers $\overset{-}{p}$, $\kappa (\neq 0)$ and $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that (4.12) holds with $\overset{-}{\underset{\sim}{F}}$ defined by (4.13). Here we have omitted $\kappa = 0$ since, by (4.13) we see that this corresponds to the shockless state $\overset{-}{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}}$.

Finally, we note that since the traction continuity condition (4.12) imposes only two scalar restrictions on the three parameters ϕ , κ and $\bar{p}$, one would anticipate that if there exists an elastostatic shock corresponding to a given $\overset{+}{\underset{\sim}{F}}$ and $\overset{+}{p}$, then in fact there exists a one – parameter family of shocks.

5.1 Weak Piecewise Homogeneous Elastostatic Shocks

We now specialize the problem posed in the general setting of Section 4.2 to the first of two simpler cases. Here we confine attention to elastostatic shocks that are weak in the sense that the departure of $\bar{\underset{\sim}{F}}$ from $\overset{+}{\underset{\sim}{F}}$ is small. Motivated by the remarks at the end of the previous section, we assume here that there exists a one-parameter family of shocks, corresponding to the given $\overset{+}{\underset{\sim}{F}}$ and $\overset{+}{p}$, depending on the parameter κ and sufficiently smooth near $\kappa = 0$. Specifically, we suppose that there are functions $\phi(\kappa)$, $\bar{p}(\kappa)$ both twice continuously differentiable in a neighborhood of $\kappa = 0$, such that $\bar{\underset{\sim}{F}}$ defined by (4.13) together with $\bar{p}(\kappa)$ conforms with the traction continuity requirement (4.12). Since from (4.13) we have that $\bar{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}}$ when $\kappa = 0$, we may use κ as a measure of the departure of $\bar{\underset{\sim}{F}}$ from $\overset{+}{\underset{\sim}{F}}$. Accordingly κ will be referred to as the shock-strength parameter.

We first record the following kinematic results which are established in [3]. Let

$$c = |\overset{+}{\underset{\sim}{F}}\underset{\sim}{L}| = |\bar{\underset{\sim}{F}}\underset{\sim}{L}| \quad . \tag{5.1}$$

Then

$$\ell_\alpha = \frac{1}{c}\overset{+}{F}_{\alpha\beta}L_\beta = \frac{1}{c}\bar{F}_{\alpha\beta}L_\beta \quad , \tag{5.2}$$

$$n_\alpha = c\overset{+}{F}{}^{-1}_{\beta\alpha}N_\beta = c\bar{F}{}^{-1}_{\beta\alpha}N_\beta \quad , \tag{5.3}$$

$$\ell_\alpha \ell_\beta + n_\alpha n_\beta = \delta_{\alpha\beta} \;, \quad L_\alpha = \epsilon_{\alpha\beta} N_\beta \;. \tag{5.4}$$

If $\phi(\varkappa)$ and $\bar{p}(\varkappa)$ exist as described above, it follows that $\underset{\sim}{L}$, $\underset{\sim}{N}$, $\underset{\sim}{\ell}$, $\underset{\sim}{n}$, $\bar{\underset{\sim}{F}}$ and c are all $\varkappa$ dependent whence we write $\underset{\sim}{L}(\varkappa)$, $\underset{\sim}{N}(\varkappa)$, $\underset{\sim}{\ell}(\varkappa)$, $\underset{\sim}{n}(\varkappa)$, $\bar{\underset{\sim}{F}}(\varkappa)$ and $c(\varkappa)$.

Because of the presumed smoothness of $\phi(\varkappa)$ we have the following Taylor expansions, where a prime denotes differentiation with respect to $\varkappa$,

$$\underset{\sim}{\ell}(\varkappa) = \underset{\sim}{\ell}(0) + \underset{\sim}{\ell}'(0)\varkappa + o(\varkappa) \;, \quad \underset{\sim}{n}(\varkappa) = \underset{\sim}{n}(0) + \underset{\sim}{n}'(0)\varkappa + o(\varkappa) \;, \tag{5.5}$$

[1]

$$\underset{\sim}{N}(\varkappa) = \underset{\sim}{N}(0) + \underset{\sim}{N}'(0)\varkappa + o(\varkappa) \;.$$

Equation (4.13) now gives

$$\bar{F}_{\alpha\beta}(\varkappa) = \overset{+}{F}_{\alpha\beta} + \varkappa \ell_\alpha(0) n_\gamma(0) \overset{+}{F}_{\gamma\beta} + o(\varkappa) \;, \tag{5.6}$$

$$\bar{F}^{-1}_{\alpha\beta}(\varkappa) = \overset{+}{F}{}^{-1}_{\alpha\beta} - \varkappa \ell_\gamma(0) n_\beta(0) \overset{+}{F}{}^{-1}_{\alpha\gamma} + o(\varkappa) \;, \tag{5.7}$$

where we have also used (5.4). This enables us to write the following Taylor expansion

$$\frac{\partial \hat{W}(\bar{\underset{\sim}{F}}(\varkappa))}{\partial F_{\alpha\beta}} = \frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} + \varkappa \frac{\partial^2 \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta} \partial F_{\gamma\delta}} \ell_\alpha(0) n_\nu(0) \overset{+}{F}_{\nu\delta} + o(\varkappa) \;. \tag{5.8}$$

The Taylor expansion of $\bar{p}(\varkappa)$ leads to

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[1] Whenever we write $o(\varkappa)$, we mean $o(\varkappa)$ as $\varkappa \to 0$.

$$\bar{p}(\kappa) = \bar{p}(0) + \kappa \bar{p}'(0) + o(\kappa) \quad . \tag{5.9}$$

Using (5.6), (5.7) and (5.9) and evaluating the traction continuity condition (4.12) to leading order, gives

$$\overset{+}{p} = \bar{p}(0) \quad . \tag{5.10}$$

Consequently we may write (5.9) as

$$\bar{p}(\kappa) = \overset{+}{p} + \kappa \bar{p}'(0) + o(\kappa) \quad . \tag{5.11}$$

We now return to the traction continuity condition (4.12) and re-evaluate it to leading order using (2.11), (5.5), (5.7), (5.8) and (5.11). This leads to

$$\left\{ c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}}) \ell_\gamma(0) n_\nu(0) \overset{+}{F}_{\nu\delta} - \bar{p}'(0) \overset{+}{F}{}^{-1}_{\beta\alpha} + \overset{+}{p} \ell_\gamma(0) n_\alpha(0) \overset{+}{F}{}^{-1}_{\beta\gamma} \right\} N_\beta(0) = 0 \quad , \tag{5.12}$$

which are two scalar equations for $\bar{p}'(0)$ and $\phi(0)$.

5.2 A Necessary Condition for the Existence of a Weak Shock

We now derive a necessary condition for equation (5.12) to have a solution $\bar{p}'(0)$, $\phi(0)$. We have from a Taylor expansion of (5.3) that

$$N_\beta(0) = \frac{1}{c(0)} \overset{+}{F}_{\nu\beta} n_\nu(0) \quad . \tag{5.13}$$

Equations (5.12) and (5.13) lead to

$$c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}}) \ell_\gamma(0) n_\pi(0) n_\nu(0) \overset{+}{F}_{\pi\delta} \overset{+}{F}_{\nu\beta} - \bar{p}'(0) n_\alpha(0) + \overset{+}{p} \ell_\gamma(0) n_\gamma(0) n_\alpha(0) = 0 \quad . \tag{5.14}$$

But since $\underset{\sim}{\ell}$ is perpendicular to $\underset{\sim}{n}$ we have $\ell_\alpha(0) n_\alpha(0) = 0$, whence (5.14) simplifies to

$$c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)n_\pi(0)n_\nu(0)\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\nu\beta} - \bar{p}'(0)n_\alpha(0) = 0 \quad . \tag{5.15}$$

Multiplying (5.15) by $n_\alpha(0)$ and making use of the fact that $\underset{\sim}{n}$ is a unit vector leads to

$$\bar{p}'(0) = c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)n_\pi(0)n_\nu(0)n_\alpha(0)\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\nu\beta} \quad . \tag{5.16}$$

Alternatively, multiplying (5.15) by $\ell_\alpha(0)$ gives

$$c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)\ell_\alpha(0)n_\pi(0)n_\nu(0)\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\nu\beta} = 0 \quad , \tag{5.17}$$

by virtue of the fact that $\underset{\sim}{\ell}\cdot\underset{\sim}{n} = 0$. Using (5.3) and (5.4) in (5.17) leads to

$$\epsilon_{\pi\nu}\epsilon_{\lambda\mu}\overset{+}{F}_{\gamma\pi}\overset{+}{F}_{\alpha\lambda}c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})N_\beta(0)N_\delta(0)N_\nu(0)N_\mu(0) = 0 \quad , \tag{5.18}$$

which because of (3.7) can be equivalently written as

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}\overset{+}{F}_{\gamma\lambda}\overset{+}{F}_{\delta\mu}Q_{\gamma\delta}(\underset{\sim}{N}(0), \overset{+}{\underset{\sim}{F}})N_\alpha(0)N_\beta(0) = 0 \quad . \tag{5.19}$$

Equation (5.19) must necessarily hold if a one parameter family of elastostatic shocks of the type being considered is to exist. On comparing with (3.18), we see that (5.19) implies a loss of ellipticity of the displacement equations of equilibrium on $\overset{+}{\Pi}$ at the given $\overset{+}{\underset{\sim}{F}}$ and $\overset{+}{p}$. We therefore have the following result:

> Theorem 1. A necessary condition for the existence of a one-parameter family of elastostatic shocks, of the kind under consideration, is that the displacement equations of equilibrium suffer a loss of ellipticity at the given deformation and pressure on $\overset{+}{\Pi}$. Furthermore, in the weak shock

limit $(\kappa \to 0)$ the material shock-line and the spatial shock-line tend respectively to a material and spatial characteristic associated with $\overset{+}{\Pi}$.

The corresponding result was obtained by Knowles and Sternberg [3] in the case of compressible elastic materials.

In the event that, corresponding to a given $\overset{+}{\underset{\sim}{F}}$ and $\overset{+}{p}$ a one-parameter family of shocks of the type being considered exists, the jumps of various physical quantities across the shock can be easily determined to leading order in terms of the given $\overset{+}{\underset{\sim}{F}}$, $\overset{+}{p}$ and the presumably determinable (from (5.12)) $\phi(0)$, $\bar{p}'(0)$. We now determine some of these jumps.

(i) <u>The jump in energy density</u> $[W]^{+}_{-}$

The Taylor expansion of $\hat{W}(\bar{\underset{\sim}{F}}(\kappa))$ about $\kappa = 0$, together with (2.9) and (5.6) leads to

$$\hat{W}(\bar{\underset{\sim}{F}}(\kappa)) = \hat{W}(\overset{+}{\underset{\sim}{F}}) + \kappa\left\{\overset{+}{\tau}_{\alpha\beta}n_{\beta}(0)\ell_{\alpha}(0) + \overset{+}{p}\ell_{\alpha}(0)n_{\alpha}(0)\right\} + o(\kappa) \quad . \tag{5.20}$$

Since $\underset{\sim}{\ell}$ is perpendicular to $\underset{\sim}{n}$ we can drop the last term in (5.20) to get

$$[W]^{-}_{+} = \kappa\, t_{\alpha}(0)\ell_{\alpha}(0) + o(\kappa) \quad , \tag{5.21}$$

where we have set

$$\underset{\sim}{t}(\kappa) = \overset{+}{\underset{\sim\sim}{\tau}}\underset{\sim}{n}(\kappa) \quad . \tag{5.22}$$

As a consequence of (2.6), (4.2), (5.3) and displacement continuity, we see immediately that $\underset{\sim}{t}(\kappa) = \overset{+}{\underset{\sim\sim}{\tau}}\underset{\sim}{n}(\kappa) = \bar{\underset{\sim\sim}{\tau}}\underset{\sim}{n}(\kappa)$ which implies the continuity of the Cauchy traction vector across $\mathcal{L}_{*}$.

(ii) <u>The stress jumps</u> $[\tau_{\alpha\beta}]^{+}_{-}$

From (2.9) we have that

$$\bar{\tau}_{\alpha\beta} = \frac{\partial \hat{W}(\bar{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} \bar{F}_{\beta\gamma} - \bar{p}\delta_{\alpha\beta} \, , \tag{5.23}$$

$$\overset{+}{\tau}_{\alpha\beta} = \frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} \overset{+}{F}_{\beta\gamma} - \bar{p}\delta_{\alpha\beta} \, , \tag{5.24}$$

which together with (2.11), (5.6), (5.8), (5.11) and (5.22) leads to

$$[\tau_{\alpha\beta}]^{-}_{+} = \varkappa\Big\{c_{\alpha\mu\gamma\delta}(\overset{+}{\underset{\sim}{F}})\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\beta\mu}\ell_{\gamma}(0)n_{\pi}(0) - \bar{p}'(0)\delta_{\alpha\beta} + \overset{+}{p}n_{\alpha}(0)\ell_{\beta}(0)$$

$$+ t_{\alpha}(0)\ell_{\beta}(0)\Big\} + o(\varkappa) \, . \tag{5.25}$$

(iii) <u>The jump in the normal stress acting on a plane perpendicular to</u> $\mathcal{L}_*$ $[\tau^{X'}_{11}]^{+}_{-}$

Consider the plane perpendicular to the spatial shock so that the normal to this plane is $\underset{\sim}{\ell}$. The jump in the normal stress acting on this plane across the shock-line, $[\tau^{X'}_{11}]^{+}_{-}$, is

$$[\tau^{X'}_{11}]^{+}_{-} = \overset{+}{\tau}_{\alpha\beta}\ell_{\alpha}\ell_{\beta} - \bar{\tau}_{\alpha\beta}\ell_{\alpha}\ell_{\beta} \, , \tag{5.26}$$

which because of (5.25) and the perpendicularity of the vectors $\underset{\sim}{\ell}$, and $\underset{\sim}{n}$ can be written to leading order as

$$[\tau^{X'}_{11}]^{+}_{-} = -\varkappa\Big\{t_{\alpha}(0)\ell_{\alpha}(0) - \bar{p}'(0)$$

$$+ c_{\alpha\mu\gamma\delta}(\overset{+}{\underset{\sim}{F}})\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\beta\mu}\ell_{\gamma}(0)\ell_{\beta}(0)\ell_{\alpha}(0)n_{\pi}(0)\Big\} + o(\varkappa) \, . \tag{5.27}$$

In view of (5.4) and (5.1[illegible]) this leads to

$$[\tau^{X'}_{11}]^+_- = -\kappa\left\{ t_\alpha(0)\ell_\alpha(0) + c_{\alpha\mu\gamma\delta}(\underset{\sim}{\bar{F}})\bar{F}_{\pi\delta}\bar{F}_{\alpha\mu}\ell_\gamma(0)n_\pi(0) - 2\bar{p}'(0)\right\} + o(\kappa) \ ,$$

which together with (5.25) gives

$$[\tau^{X'}_{11}]^+_- = -\,\mathrm{tr}\,\underset{\sim}{\bar{\tau}}'(0) - \kappa\bar{p}'(0) + o(\kappa) \ . \tag{5.28}$$

6.1 Finite Elastostatic Shocks in Isotropic Incompressible Materials

We now return to shocks of finite strength, but assume the material at hand to be isotropic. Substituting (2.31) in (4.12) and making use of (2.4) and (5.3) leads to

$$\left\{2W'(\overset{+}{I})\overset{+}{G}_{\alpha\beta}-\overset{+}{p}\delta_{\alpha\beta}\right\}n_\beta=\left\{2W'(\bar{I})\bar{G}_{\alpha\beta}-\bar{p}\delta_{\alpha\beta}\right\}n_\beta \ , \tag{6.1}$$

where

$$\overset{+}{I}=\overset{+}{F}_{\alpha\beta}\overset{+}{F}_{\alpha\beta}=\overset{+}{G}_{\alpha\alpha} \ , \quad \bar{I}=\bar{F}_{\alpha\beta}\bar{F}_{\alpha\beta}=\bar{G}_{\alpha\alpha} \ . \tag{6.2}$$

Clearly, (6.1) is simply a statement of the fact that the Cauchy traction is continuous across the spatial shock. The original problem concerning the existence of elastostatic shocks can now be posed as follows: given a constant tensor $\overset{+}{\underset{\sim}{F}}$ with unit determinant and a real constant $\overset{+}{p}$, determine real numbers $\bar{p}$, $\varkappa(\neq 0)$ and ϕ such that (6.1) holds with $\bar{\underset{\sim}{G}}$ given by (4.13), (6.2).

If we express (6.1) in terms of its components in the frame X', we have

$$2\overset{+}{G}{}^{X'}_{12}\,W'(\overset{+}{I})=2\bar{G}{}^{X'}_{12}\,W'(\bar{I}) \ , \tag{6.3}$$

$$2\overset{+}{G}{}^{X'}_{22}\,W'(\overset{+}{I})-\overset{+}{p}=2\bar{G}{}^{X'}_{22}\,W'(\bar{I})-\bar{p} \ . \tag{6.4}$$

As observed earlier, (6.3) and (6.4) impose only two scalar restrictions on the three quantities ϕ, $\varkappa$ and $\bar{p}$. Furthermore since $\bar{p}$ enters

only in (6.4), and there too only linearly, we may consider (6.3) and (6.4) separately i.e. if there are numbers κ and ϕ such that (6.3) holds, then there certainly is a third number $\bar{p}$ such that (6.4) holds as well. The existence of an elastostatic shock therefore depends on whether there are numbers κ and ϕ such that (6.3) holds.

To pursue this question further, we need the components of $\overset{+}{\underset{\sim}{G}}$ and $\overset{-}{\underset{\sim}{G}}$ in the frame X'. With no loss of generality let us take X to be a principal frame for $\overset{+}{\underset{\sim}{G}}$. Then

$$\overset{+}{\underset{\sim}{G}}{}^{X} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}, \quad \lambda_1\lambda_2 = 1 \, . \tag{6.5}$$

By the change of frame formula for tensors we have

$$\overset{+}{\underset{\sim}{G}}{}^{X'} = \underset{\sim}{R}\,\overset{+}{\underset{\sim}{G}}{}^{X}\underset{\sim}{R}^{T} , \tag{6.6}$$

$$\underset{\sim}{R} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} . \tag{6.7}$$

Equations (6.5) - (6.7) lead to

$$\overset{+}{\underset{\sim}{G}}{}^{X'} = \begin{bmatrix} \dfrac{\lambda_1^2+\lambda_2^2}{2} + \dfrac{\lambda_1^2-\lambda_2^2}{2}\cos 2\phi & -\dfrac{\lambda_1^2-\lambda_2^2}{2}\sin 2\phi \\ -\dfrac{\lambda_1^2-\lambda_2^2}{2}\sin 2\phi & \dfrac{\lambda_1^2+\lambda_2^2}{2} - \dfrac{\lambda_1^2-\lambda_2^2}{2}\cos 2\phi \end{bmatrix} . \tag{6.8}$$

If we now set

$$\beta = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2} , \tag{6.9}$$

we can write (6.8) because of the second of (6.5) as

$$\overset{+}{\underset{\sim}{G}}{}^{X'} = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1+\beta\cos 2\phi & -\beta\sin 2\phi \\ -\beta\sin 2\phi & 1-\beta\cos 2\phi \end{bmatrix} . \tag{6.10}$$

It is clear from (6.5) and (6.9), that the value of β alone suffices to determine $\overset{+}{\underset{\sim}{G}}{}^{X}$ completely, and in this sense <u>β is a measure of the deformation on $\overset{+}{\Pi}$.</u> Note that because $\lambda_\alpha > 0$, (6.9) implies that

$$1 > \beta > -1 . \tag{6.11}$$

Furthermore, we have $\beta = 0$ if and only if the part of the body occupying $\overset{+}{\Pi}$ in its undeformed configuration remains undeformed[1] under the mapping (4.7).

We now find from (4.14), (6.2) and (6.10) that

$$\overset{=}{\underset{\sim}{G}}{}^{X'} = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1+\beta\cos 2\phi - 2\kappa\beta\sin 2\phi + \kappa^2(1-\beta\cos 2\phi) & -\beta\sin 2\phi + \kappa(1-\beta\cos 2\phi) \\ -\beta\sin 2\phi + \kappa(1-\beta\cos 2\phi) & 1-\beta\cos 2\phi \end{bmatrix} , \tag{6.12}$$

[1] $\overset{+}{\underset{\sim}{F}}$ is then a proper orthogonal tensor.

and from (6.2), (6.10) and (6.12) that

$$\overset{+}{I} = \frac{2}{\sqrt{1-\beta^2}} \, , \tag{6.13}$$

$$\bar{I} = \frac{2 - 2\varkappa\beta\sin 2\phi + \varkappa^2(1-\beta\cos 2\phi)}{\sqrt{1-\beta^2}} \, . \tag{6.14}$$

Returning to the traction continuity requirement (6.3) with (6.10), (6.12) - (6.14) we find

$$-\beta\sin 2\phi W'\left(\frac{2}{\sqrt{1-\beta^2}}\right) = \Big\{-\beta\sin 2\phi$$

$$+\varkappa(1-\beta\cos 2\phi)\Big\} W'\left(\frac{2 - 2\varkappa\beta\sin 2\phi + \varkappa^2(1-\beta\cos 2\phi)}{\sqrt{1-\beta^2}}\right) \, . \tag{6.15}$$

We may now pose the problem as follows: given a number β in $(-1, 1)$, find numbers $\varkappa \neq 0$ and ϕ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that (6.15) holds.

If, for the given β in $(-1, 1)$ and any ϕ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ there does <u>not</u> exist a root $\varkappa \neq 0$ to (6.15), the material is incapable of sustaining an elastostatic shock corresponding to the given deformation associated with β on $\overset{+}{\Pi}$. On the other hand if, for the given β in $(-1, 1)$ and for some ϕ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ there is a root $\varkappa \neq 0$ to equation (6.15), then there exists a corresponding elastostatic shock. Therefore, we now investigate the possibility that (6.15) has a root $\varkappa \neq 0$ for all values of ϕ and β such that $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$, $|\beta| < 1$.

Finally we observe from (6.15) that if for some pair (ϕ, β),

$|\beta| < 1$ and $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$, there exists a root $\varkappa$ to equation (6.15), then

(i) $-\varkappa$ is a root of (6.15) for the values $(-\phi, \beta)$, and

(ii) $-\varkappa$ is a root of (6.15) for the values $(\frac{\pi}{2} - \phi, -\beta)$.

It therefore follows that, as far as the issue of existence is concerned, we may in fact restrict ϕ to $\left[0, \frac{\pi}{2}\right]$ and β to $[0, 1)$. If we define the set $\mathfrak{G}$ by

$$\mathfrak{G} = \{(\phi, \beta) \,|\, 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \beta < 1\}, \tag{6.16}$$

we need to look at the question of the existence of a root $\varkappa \neq 0$ to equation (6.15) for every (ϕ, β) in $\mathfrak{G}$.

6.2 Some General Results

We now establish some general results concerning the existence of elastostatic shocks, valid for a homogeneous, isotropic, incompressible elastic solid which has a positive shear modulus.[1]

We first make the following preliminary observation. If $\beta = 0$ or $\phi = 0$ or $\phi = \frac{\pi}{2}$, the only root of (6.15) is $\varkappa = 0$. This follows directly from (6.15) because of (2.40). Consequently, for a material of the type we are considering, no elastostatic shock is possible if the part of the body occupying $\overset{+}{\Pi}_*$ is undeformed, nor can any spatial shock-line be inclined at 0 or $\frac{\pi}{2}$ to the principal axes of $\overset{+}{\underset{\sim}{G}}$. We may now restrict attention to the interior $\overset{\circ}{\mathfrak{G}}$ of the set $\mathfrak{G}$:

$$\overset{\circ}{\mathfrak{G}} = \{(\phi, \beta) \,|\, 0 < \phi < \frac{\pi}{2}, \quad 0 < \beta < 1\}. \tag{6.17}$$

If we set

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[1] We assume from here on that (2.40) holds.

$$b = b(\phi, \beta) = -\frac{\beta \sin 2\phi}{(1-\beta^2)^{\frac{1}{2}}} \quad \text{on } \mathring{G}\ , \tag{6.18}$$

$$c = c(\phi, \beta) = \frac{(1-\beta\cos 2\phi)}{(1-\beta^2)^{\frac{1}{2}}} \quad \text{on } \mathring{G}\ , \tag{6.19}$$

we can write (6.15) using (6.13), (6.18) and (6.19) as

$$bW'(\overset{+}{I}) = (b + c\kappa)W'(\overset{+}{I} + 2b\kappa + c\kappa^2)\ . \tag{6.20}$$

Clearly

$$b < 0 \quad \text{on } \mathring{G}\ , \tag{6.21}$$

$$c > 0 \quad \text{on } \mathring{G}\ . \tag{6.22}$$

Choose and <u>fix</u> a point $(\hat{\phi}\ , \hat{\beta})$ in $\mathring{G}$. At this fixed value of ϕ and β we define the function h by

$$h(x) = (b + cx)W'(\overset{+}{I} + 2bx + cx^2) - bW'(\overset{+}{I}) \quad \text{for } |x| < \infty\ , \tag{6.23}$$ [1]

where $\overset{+}{I}$, b and c are given by (6.13), (6.18) and (6.19) respectively evaluated at $(\hat{\phi}\ , \hat{\beta})$. If the plane strain elastic potential W(I) is twice continuously differentiable on $I \geq 2$, as we have implicitly assumed, it follows that h(x) is continuously differentiable on $(-\infty\ , \infty)$. If there exists an equilibrium shock corresponding to the homogeneous deformation associated with $\hat{\beta}$ on $\overset{+}{\Pi}$ and inclined at an angle $\hat{\phi}$ to the y_1-axis, it is necessary and sufficient that h(x) have a zero at some

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[1] From (6.13), (6.18), (6.19) and (6.22) we have that
$\overset{+}{I} + 2bx + cx^2 = 2 + \frac{(b+cx)^2}{c} + \frac{(c-1)^2}{c} \geq 2$ for all $|x| < \infty$.

$x \neq 0$. The zero of $h(x)$ gives the shock strength $\varkappa$.

Because of (2.40), (6.21) and (6.22) we see from (6.23) that

$$h(0) = 0 \quad , \tag{6.24}$$

$$h(-\frac{b}{c}) = -bW'(\overset{+}{I}) > 0 \quad , \quad -\frac{b}{c} > 0 \quad . \tag{6.25}$$

It now follows from (6.24), (6.25) and the smoothness of $h(x)$ that:

(i) if $h'(0) < 0$, then there exists a zero of $h(x)$ other than $x = 0$,

(ii) if there exists a zero other than $x = 0$ of $h(x)$, then there exists a number $\varkappa_* \neq 0$ such that $h'(\varkappa_*) = 0$. Furthermore, since $h'(-b/c) = cW'(\overset{+}{I} - b^2/c) > 0$ we have

$$\varkappa_* \neq -\frac{b}{c} \quad . \tag{6.26}$$

Because of the remarks made before (6.24), we may interpret (i) and (ii) as follows:

(a) $h'(0) < 0$ is sufficient to ensure the existence of an elastostatic shock corresponding to the deformation associated with $\hat{\beta}$ on $\overset{+}{\Pi}$ with spatial shock inclination $\hat{\phi}$.

(b) If an elastostatic shock as just described is to exist, then it is necessary that $h'(\varkappa_*) = 0$ for some number $\varkappa_*$.

This leads to the main results of this section which we now establish. We first introduce the following terminology. Recall from (2.40) and (3.21), that a loss of ellipticity of the displacement equations of equilibrium can occur at some deformation and at some point if and only if

$$\frac{2W''(I)}{W'(I)}(I-2)+1 \leq 0 \quad . \tag{6.27}$$

If ellipticity is lost because in fact the strict inequality holds in (6.27), we say that a strict failure of ellipticity has occurred. Since four characteristic curves exist in this case (see Section 3.3) one may say that the displacement equations of equilibrium are hyperbolic at such a deformation on $\overset{+}{\Pi}$.

Theorem 2. A strict failure of ellipticity of the displacement equations of equilibrium, at the given homogeneous deformation and pressure on $\overset{+}{\Pi}$, is sufficient to ensure the existence of a corresponding elastostatic shock in a homogeneous, isotropic, incompressible, elastic solid with a positive shear modulus.[1]

Proof:

By hypothesis, the given deformation gradient $\overset{+}{\underset{\sim}{F}}$ is such that the associated value of β , say $\hat{\beta}$, given by (2.4), (6.5) and (6.9) conforms with the inequality

$$\frac{2W''(\overset{+}{I})}{W'(\overset{+}{I})}(\overset{+}{I}-2)+1<0 \quad , \tag{6.28}$$

where by (6.13)

$$\overset{+}{I} = \frac{2}{\sqrt{1-\hat{\beta}^2}} \quad . \tag{6.29}$$

Note from (2.40) and (6.28) that necessarily $\overset{+}{I} \neq 2$, whence $\hat{\beta} \neq 0$. We now choose the value of $\hat{\phi}$ as

[1] This and the following result can be readily modified for materials whose shear modulus is not always positive.

$$\hat{\phi} = \frac{1}{2}\cos^{-1}\left(\frac{1-\sqrt{1-\hat{\beta}^2}}{\hat{\beta}}\right) \quad . \tag{6.30}$$[1]

We will show that corresponding to the given homogeneous deformation on $\overset{+}{\Pi}$, such that the associated value of β conforms with (6.28) and (6.29), there exists an elastostatic shock at the inclination $\hat{\phi}$ given by (6.30).

According to (6.23) and statement (a) following (6.26), we need only show that

$$2b^2W''(\overset{+}{I}) + cW'(\overset{+}{I}) < 0 \quad \text{at } (\hat{\phi}, \hat{\beta}) \; , \tag{6.31}$$

in order to establish this. Using (2.40) we may write (6.28) alternately as

$$2b^2W''(\overset{+}{I}) + cW'(\overset{+}{I}) + \left(\frac{b^2}{\overset{+}{I}-2} - c\right)W'(\overset{+}{I}) < 0 \quad , \tag{6.32}$$

where b and c are defined by (6.18), (6.19) and evaluated at $(\hat{\phi}, \hat{\beta})$. Using (6.18), (6.19) and (6.29) in (6.32) we find

$$2b^2W''(\overset{+}{I}) + cW'(\overset{+}{I}) - \frac{\hat{\beta}^2}{(1-\hat{\beta}^2)(\overset{+}{I}-2)}\left\{\cos 2\hat{\phi} - \frac{1-\sqrt{1-\hat{\beta}^2}}{\hat{\beta}}\right\}^2 W'(\overset{+}{I}) < 0 \quad , \tag{6.33}$$

which because of (6.30) reduces to (6.31), which in turn establishes our result.

<u>Theorem 3</u>. A <u>necessary</u> condition for the existence of a piecewise homogeneous elastostatic shock in a homogeneous, isotropic, incompressible, elastic

[1]Since $\hat{\beta} \neq 0$, $|\hat{\beta}| < 1$ this defines a real angle $\hat{\phi}$ in $(0, \frac{\pi}{2})$.

solid with positive shear modulus, is that the displacement equations of equilibrium suffer a loss of ellipticity at <u>some</u> homogeneous deformation.

<u>Proof</u>:

By hypothesis there is a point $(\hat{\phi}, \hat{\beta})$ in $\mathring{\mathcal{a}}$ such that there exists an associated elastostatic shock. By statement (b) following (6.26) then, there is a real number κ_* such that

$$h'(\kappa_*) = 0 \tag{6.34}$$

where $h(x)$ is given by (6.23) with b, c and $\overset{+}{I}$ evaluated at $(\hat{\phi}, \hat{\beta})$. Equations (6.23) and (6.34) give that

$$cW'(\overset{+}{I} + 2b\kappa_* + c\kappa_*^2) + 2(b + c\kappa_*)^2 W''(\overset{+}{I} + 2b\kappa_* + c\kappa_*^2) = 0 \quad . \tag{6.35}$$

Let

$$I_* = \overset{+}{I} + 2b\kappa_* + c\kappa_*^2 \quad , \tag{6.36}$$

so that we have from (6.35) that

$$(I_* - 2)\left\{cW'(I_*) + 2(b + c\kappa_*)^2 W''(I_*)\right\} = 0 \quad .$$

It follows from this that

$$(I_* - 2)\left\{cW'(I_*) + 2(b + c\kappa_*)^2 W''(I_*)\right\}$$

$$\leq \frac{\hat{\beta}^2}{(1 - \hat{\beta}^2)} \left\{\cos 2\hat{\phi} - \frac{1 - \sqrt{1 - \hat{\beta}^2}}{\hat{\beta}}\right\}^2 W'(I_*) \quad , \tag{6.37}$$

since by virtue of (2.40) and (6.11) the right hand side of (6.37) is

non-negative. Using (6.13), (6.18), (6.19) and (6.36) in (6.37) leads to

$$(b + c\kappa_*)^2\left\{2W''(I_*)(I_* - 2) + W'(I_*)\right\} \leq 0 \ , \tag{6.38}$$

which because of (2.40) and (6.26) gives

$$\frac{2W''(I_*)}{W'(I_*)}(I_* - 2) + 1 \leq 0 \ . \tag{6.39}$$

This implies a loss of ellipticity of the displacement equations of equilibrium at a homogeneous deformation in which the deformation gradient $\underset{\sim}{F}$ is such that $F_{\alpha\beta}F_{\alpha\beta} = I_*$.

To summarize, we have shown that for the type of material at hand, a strict loss of ellipticity at the given deformation is sufficient to ensure the existence of a corresponding elastostatic shock. On the other hand, a loss of ellipticity at some homogeneous deformation is necessary, if an elastostatic shock is to exist.

We draw attention to the fact that Theorem 2 does <u>not</u> imply that if ellipticity is strictly lost at the given deformation then the corresponding configuration of the body must involve a shock. Rather, it claims that such a configuration is available. There is also a shockless configuration available corresponding to the root $\kappa = 0$ of (6.20). Likewise, a loss of ellipticity at the given deformation is <u>not necessary</u> for a corresponding elastostatic shock to exist. In a boundary-value problem that we have studied, the results of which will be reported in a separate paper, we encountered configurations of a body involving elastostatic shocks such that the displacement equations of equilibrium were elliptic on <u>both</u> sides of the shock-line.

7.1 Dissipativity Inequality

If we admit weak solutions into the discussion of a problem, (such as those of the type introduced in Sections 4.0 - 6.0), we would anticipate that since the admissible class of solutions has been greatly widened, there could possibly be many solutions to that problem. It is well known that this is indeed the case in the theory of quasi-linear hyperbolic partial differential equations. See for example Lax [8]. The boundary-value problem referred to at the end of the preceding section confirms this to be the case in the present context as well.

In such circumstances, it is essential to introduce criteria which single out a physically admissible solution from among the many solutions admitted by the differential equations. The second law of thermodynamics appears to play such a role in gas dynamics. Lax [8] has examined "entropy conditions" which furnish such criteria in the context of hyperbolic systems of conservation laws.

An analogous "entropy condition" in the context of elastostatics was proposed by Knowles and Sternberg [3] and subsequently extended by Knowles [4]. A thermodynamic motivation for the proposed condition, in the case of compressible materials, was also given in [4]. In the three-dimensional case, a quasi-static time dependent family of equilibrium states was considered, the time merely playing the role of a history parameter, and it was then required that

$$\int_{\partial \mathcal{D}} \underset{\sim}{\sigma} \underset{\sim}{N} \cdot \underset{\sim}{v} \, dA - \frac{d}{dt} \int_{\mathcal{D}} \hat{W}(\underset{\sim}{F}) dv \geq 0 \qquad (7.1)^1$$

for every regular sub-domain $\mathcal{D}$ of $\mathcal{R}$, at each instant of the time interval considered. Here t is the time and $\underset{\sim}{v}$ the quasi-static particle velocity. Equation (7.1) gives expression to the idea that the rate at which elastic energy is being stored in $\mathcal{D}$ cannot exceed the rate at which work is being done on $\mathcal{D}$.

One shows easily that for a sub-domain $\mathcal{D}$ of the body which is such that the field quantities have classical smoothness properties at each interior point, the global condition (7.1) holds with inequality replaced by equality by virtue of the field equations. This is indeed as one would expect, and accordingly (7.1) imposes no local restrictions at a point where the fields are smooth. If however an elastostatic shock is present in the domain $\mathcal{D}$, then (7.1) does <u>not</u> hold automatically and consequently, at each point on the shock it imposes a local restriction on the jumps of the field quantities.

Now consider a quasi-static family of plane strain piecewise homogeneous elastostatic shocks in a homogeneous, incompressible, elastic solid. It can be shown that, if at some instant t (7.1) holds with strict inequality for all sub-domains $\mathcal{D}$ which intersect $\mathcal{L}$, then

(i) the motion of the shock-line $\mathcal{L}$ at that instant is translatory in a direction not parallel to itself. Moreover, if we orient the shock-line $\mathcal{L}$ such that this translation is directed into $\overset{+}{\Pi}$, then at that instant

[1] Body forces were omitted from this discussion.

$$[\hat{W}(\underset{\sim}{F}) - \sigma_{\alpha\beta}N_\beta F_{\alpha\gamma}N_\gamma]^+_- > 0 \quad . \tag{7.2}$$

Conversely, if at some time t the quasi-static family of solutions conforms with (i), then (7.1) holds with strict inequality at that instant.

On the other hand we can show that if at the instant t (7.1) holds with equality for all sub-domains $\mathcal{D}$ then either

(ii) the shock-line $\mathcal{L}$ is instantaneously stationary at that moment,

or (iii) the shock-line $\mathcal{L}$ is instantaneously in a state of translation parallel to itself at that moment,

or (iv) the jump of $\hat{W}(\underset{\sim}{F}) - \sigma_{\alpha\beta}N_\beta F_{\alpha\gamma}N_\gamma$ across the shock is zero, (in which case the shock-line motion is not restricted to being translatory).

Conversely, if at some time t the quasi-static family of solutions conforms with one of (ii), (iii), and (iv), then (7.1) holds with equality at that instant.

Finally one can show that if at some time t (7.1) holds for all sub-domains $\mathcal{D}$, and if in addition it holds with equality for <u>some</u> sub-domain which intersects the shock, then in fact, at that instant (7.1) holds with equality for <u>all</u> sub-domains $\mathcal{D}$. We conclude from this that the preceding are the only possibilities. Therefore, if (7.1) holds it is <u>necessary</u> that one of (i) - (iv) hold. Conversely if one of (i) - (iv) holds this is <u>sufficient</u> to ensure that (7.1) hold.

One arrives at (i) - (iv) by applying to the incompressible case the parallel arguments used by Knowles and Sternberg in [3], or by specializing to this context the results of Knowles [4]. Since (7.1) implies that the presence of an elastostatic shock decreases, or at

least does not increase, the stored energy in the body, we refer to (7.2) as the dissipativity inequality.

It is apparent from (i) - (iv) that the dissipativity requirement (7.1) may be viewed as restricting the admissible class of quasi-static motions. The only quasi-static motions admitted by it are those in which the value of $\{-\hat{W}(\underset{\sim}{F}) + \underset{\sim}{F}\,\underset{\sim}{N}\cdot\underset{\sim}{\sigma}\,\underset{\sim}{N}\}$ at a particle does not decrease as the particle crosses the shock-line.

It may be remarked that the dissipativity inequality does not rule out any piecewise homogeneous elastostatic shock itself as being inadmissible, since any given elastostatic shock can always be embedded in a suitable time dependent family of such shocks which conforms with the dissipativity inequality.

As one would expect, and as is verified by Knowles [4], these results remain true locally in the general case of a curved shock in a non-homogeneous elastic field, with the exception that the shock motion may no longer be restricted to translation. The latter property is clearly peculiar to piecewise homogeneous elastostatic shocks.

Using (2.4), (2.6), (4.13), (5.3), and evaluating the left hand side of (7.2) in the frame X' leads to

$$[\hat{W}(\underset{\sim}{F}) - F_{\alpha\beta}\sigma_{\alpha\gamma}N_{\beta}N_{\gamma}]^{+}_{-} = [\hat{W}(\underset{\sim}{F})]^{+}_{-} + \kappa\tau^{X'}_{12} , \tag{7.3}$$

where we have also used the fact that $\tau^{X'}_{12}$ is continuous across $\mathcal{L}_{*}$. Therefore the inequality (7.2) may be written in the simpler form

$$[\hat{W}(\underset{\sim}{F})]^{+}_{-} + \kappa\tau^{X'}_{12} > 0 , \tag{7.4}$$

where $\mathcal{L}$ is presumed to be oriented such that it moves into $\overset{+}{\Pi}$ as time

t increases.

In the particular case when the material at hand is isotropic, we have from (2.33), (6.10) and (6.18) that

$$\tau^{X'}_{12} = 2bW'(\overset{+}{I}) \ , \tag{7.5}$$

whence (7.4), by virtue of (6.14), (6.18) and (6.19), may be equivalently written as

$$W(\overset{+}{I}) - W(\overset{+}{I} + 2b\varkappa + c\varkappa^2) + 2b\varkappa W'(\overset{+}{I}) > 0 \ . \tag{7.6}$$

Note, however, that (6.23) may alternatively be written as

$$h(x) = \frac{1}{2}\frac{\partial W}{\partial x}(\overset{+}{I} + 2bx + cx^2) - bW'(\overset{+}{I}) \ , \tag{7.7}$$

whence (7.6) takes the simple form

$$\int_0^{\varkappa} h(x)dx < 0 \ . \tag{7.8}$$

We will make use of this form of the dissipativity inequality in the example taken up in the next section.

Finally, we return to <u>anisotropic</u>, incompressible, elastic solids in order to determine the weak shock approximation to the value of the jump of $\{\hat{W}(\underset{\sim}{F}) - F_{\alpha\beta}\sigma_{\alpha\gamma}N_\beta N_\gamma\}$ across the shock-line. Recall from Section 5.1, where we first looked at weak elastostatic shocks, that we now assume that, given the deformation gradient $\overset{+}{\underset{\sim}{F}}$ with unit determinant and the pressure $\overset{+}{p}$, there exist functions $\phi(\varkappa)$ and $\bar{p}(\varkappa)$, both sufficiently smooth in a neighborhood of $\varkappa = 0$, such that $\bar{F}_{\alpha\beta}(\varkappa)$

defined by

$$\bar{F}_{\alpha\beta}(\kappa) = \overset{+}{F}_{\alpha\beta} + \kappa \ell_{\alpha}(\kappa) n_{\gamma}(\kappa) \overset{+}{F}_{\gamma\beta} \ , \tag{7.9}$$

conforms with the traction continuity requirement (4.12). Observe from (7.9) that

$$\bar{F}^{-1}_{\alpha\beta}(\kappa) = \overset{+}{F}{}^{-1}_{\alpha\beta} - \kappa \ell_{\gamma}(\kappa) n_{\beta}(\kappa) \overset{+}{F}{}^{-1}_{\alpha\gamma} \ . \tag{7.10}$$

It is first necessary to analyze the traction continuity condition (4.12). To this end set

$$\Delta_{\alpha}(\kappa) = \overset{+}{\sigma}_{\alpha\beta} N_{\beta}(\kappa) - \bar{\sigma}_{\alpha\beta}(\kappa) N_{\beta}(\kappa) \ , \tag{7.11}$$

which because of (4.9), (5.3), (7.10) and the perpendicularity of $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$ leads to

$$\Delta_{\alpha}(\kappa) = \left\{ \frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \frac{\partial \hat{W}(\bar{\underset{\sim}{F}}(\kappa))}{\partial F_{\alpha\beta}} \right\} N_{\beta}(\kappa) + (\bar{p}(\kappa) - \overset{+}{p}) \overset{+}{F}{}^{-1}_{\beta\alpha} N_{\beta}(\kappa) \ . \tag{7.12}$$

Differentiating (7.12) with respect to κ and using (2.11), (5.10) and (7.9) gives

$$\Delta'_{\alpha}(0) = - c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}}) \overset{+}{F}_{\pi\delta} N_{\beta}(0) \ell_{\gamma}(0) n_{\pi}(0) + \bar{p}'(0) \overset{+}{F}{}^{-1}_{\beta\alpha} N_{\beta}(0) \ . \tag{7.13}$$

The continuity of traction across the shock requires that

$$\Delta_{\alpha}(\kappa) = 0 \tag{7.14}$$

for all sufficiently small κ, from which it follows that in particular

$$\Delta'_{\alpha}(0) = 0 \ . \tag{7.15}$$

From (7.13) and (7.15) we find that

$$c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\overset{+}{F}_{\pi\delta}\ell_\gamma(0)n_\pi(0)N_\beta(0) - \bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\alpha}N_\beta(0) = 0 \quad . \tag{7.16}$$

As one would expect, (7.16) is in fact the same as (5.12) because of (5.3). Differentiation of (7.12) twice with respect to $\varkappa$, together with the symmetry $c_{\alpha\beta\gamma\delta} = c_{\gamma\delta\alpha\beta}$, the fact that $\underset{\sim}{\ell}\cdot\underset{\sim}{n} = 0$ and (2.11), (5.1) - (5.3), (5.10) and (7.9) leads to

$$\begin{aligned}\ell_\alpha(0)\Delta''_\alpha(0) = &- c^2(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)\ell_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0)\\ &- 4c(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)N_\beta(0)N'_\delta(0)\\ &- 2\bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\gamma}N_\beta(0)\ell'_\alpha(0) + 2\bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\alpha}N'_\beta(0)\ell_\alpha(0) \quad ,\end{aligned} \tag{7.17}$$

where we have set

$$d_{\alpha\beta\gamma\delta\lambda\mu}(\underset{\sim}{F}) = \frac{\partial^3 \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}\partial F_{\lambda\mu}} \quad , \tag{7.18}$$

and $c(\varkappa)$ was defined in (5.1). Because of (7.14) we have that $\Delta''_\alpha(0) = 0$, whence we have from (7.17) that

$$\begin{aligned}&4c(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)N_\beta(0)N'_\delta(0)\\ &= - c^2(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)\ell_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0)\\ &\quad - 2\bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\gamma}N_\beta(0)\ell'_\alpha(0) + 2\bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\alpha}N'_\beta(0)\ell_\alpha(0) \quad .\end{aligned} \tag{7.19}$$

We now compute the jump in $\{\hat{W}(\underset{\sim}{F}) - \underset{\sim}{F}\underset{\sim}{N}\cdot\underset{\sim}{\sigma}\underset{\sim}{N}\}$ across the shock.

To this end, let

$$\eta(\varkappa) = \hat{W}(\bar{\underset{\sim}{F}}(\varkappa)) - \hat{W}(\overset{+}{\underset{\sim}{F}}) + \overset{+}{F}_{\alpha\beta}N_{\beta}(\varkappa)\overset{+}{\sigma}_{\alpha\gamma}N_{\gamma}(\varkappa)$$

$$- \bar{F}_{\alpha\beta}(\varkappa)N_{\beta}(\varkappa)\bar{\sigma}_{\alpha\gamma}(\varkappa)N_{\gamma}(\varkappa) \quad . \tag{7.20}$$

Because of traction continuity, the fact that $\underset{\sim}{\ell} \cdot \underset{\sim}{n} = 0$, (4.9), (5.3) and (7.9) we can write (7.20) as

$$\eta(\varkappa) = \hat{W}(\bar{\underset{\sim}{F}}(\varkappa)) - \hat{W}(\overset{+}{\underset{\sim}{F}}) - \varkappa c(\varkappa)\ell_{\alpha}(\varkappa)N_{\gamma}(\varkappa)\frac{\partial \hat{W}(\bar{\underset{\sim}{F}}(\varkappa))}{\partial F_{\alpha\gamma}} \quad . \tag{7.21}$$

Clearly,

$$\eta(0) = 0 \quad , \tag{7.22}$$

by virtue of (7.9). Differentiating (7.21) with respect to $\varkappa$ and using (2.11), (5.3) and (7.9) gives

$$\eta'(\varkappa) = -\varkappa c(\varkappa)c_{\alpha\beta\gamma\delta}(\bar{\underset{\sim}{F}}(\varkappa))\ell_{\gamma}(\varkappa)N_{\delta}(\varkappa)\frac{d}{d\varkappa}\{\varkappa c(\varkappa)\ell_{\alpha}(\varkappa)N_{\beta}(\varkappa)\} \quad , \tag{7.23}$$

from which we have that

$$\eta'(0) = 0 \quad . \tag{7.24}$$

Differentiating (7.23) with respect to $\varkappa$ and using (7.9) leads to

$$\eta''(0) = -c^{2}(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_{\alpha}(0)\ell_{\gamma}(0)N_{\beta}(0)N_{\delta}(0) \quad , \tag{7.25}$$

which because of (5.3) and (7.16) gives

$$\eta''(0) = -\bar{p}'(0)n_{\alpha}(0)\ell_{\alpha}(0) \quad , \tag{7.26}$$

which in turn, because $\ell_\alpha(0)n_\alpha(0)=0$, implies that

$$\eta''(0)=0 \quad . \tag{7.27}$$

Finally, differentiating (7.23) twice with respect to $\varkappa$, using the symmetry $c_{\alpha\beta\gamma\delta}=c_{\gamma\delta\alpha\beta}$, (5.3), (7.9) and (7.18) leads to

$$\begin{aligned}\eta'''(0) = &- 2c^3(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)\ell_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0)\\ &- 6c^2(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)N'_\delta(0)N_\beta(0)\\ &- 6c(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)N_\beta(0)N_\delta(0)\{c'(0)\ell_\alpha(0)+c(0)\ell'_\alpha(0)\}\end{aligned} \tag{7.28}$$

which on using (5.3), (7.19) and $\ell_\alpha(0)n_\alpha(0)=0$ implies that

$$\begin{aligned}\eta'''(0) = &- \tfrac{1}{2}c^3(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)\ell_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0)\\ &- 3\bar{p}'(0)\{n'_\alpha(0)\ell_\alpha(0)+n_\alpha(0)\ell'_\alpha(0)\}\end{aligned} \tag{7.29}$$

Since the vectors $\underset{\sim}{\ell}(\varkappa)$ and $\underset{\sim}{n}(\varkappa)$ are perpendicular to each other,

$$\ell_\alpha(\varkappa)n_\alpha(\varkappa)=0 \quad \text{for all sufficiently small } \varkappa \quad . \tag{7.30}$$

Differentiating (7.30) with respect to $\varkappa$ shows that

$$\ell'_\alpha(0)n_\alpha(0)+\ell_\alpha(0)n'_\alpha(0)=0 \quad , \tag{7.31}$$

so that finally we may write (7.29) as

$$\eta'''(0) = - \tfrac{1}{2}c^3(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)\ell_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0) \quad . \tag{7.32}$$

Therefore (7.21), (7.22), (7.24) and (7.27) allow us to write

$$[\hat{W}(\underset{\sim}{F}) - F_{\alpha\beta}N_{\beta}\sigma_{\alpha\gamma}N_{\gamma}]_{+}^{-} = \frac{1}{6}\eta'''(0)\varkappa^{3} + o(\varkappa^{3}) \quad \text{as} \quad \varkappa \to 0 \ , \tag{7.33}$$

where $\eta'''(0)$ is given by (7.32). We observe that the jump in $\{\hat{W}(\underset{\sim}{F}) - F_{\alpha\beta}N_{\beta}\sigma_{\alpha\gamma}N_{\gamma}\}$ across the shock is of the third order in the shock strength $\varkappa$, which is as in the case of compressible elastic solids. This is analogous to the situation in gas dynamics where the entropy jump is of the third order in the appropriate shock strength.[1]

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[1]See references cited in [4].

8.1 An Illustrative Example

For the purpose of illustrating the results of the previous sections and demonstrating how in a particular case one could in fact obtain even more information than has been indicated, we now specialize our constitutive law. Consider the hypothetical class of homogeneous, *isotropic, incompressible, elastic solids for which the plane strain* elastic potential is given by

$$W(I) = \mu k_0^2 \left\{ 1 - \exp\left(- \frac{(I-2)}{2k_0^2} \right) \right\} \quad , \quad \mu > 0 \; , \qquad k_0 > 0 \; . \tag{8.1}$$

One sees immediately from (8.1) that (2.40) is satisfied whence this class of materials has a positive shear modulus.

According to (3.32), we have in simple shear, the shear stress-amount of shear relation

$$\tau(k) = \mu k \exp\left(- \frac{k^2}{2k_0^2} \right) \; . \tag{8.2}$$

A sketch of the response curve in shear defined by (8.2) is shown in Fig. 2. The significant feature of this for our purposes is that $\tau'(k)$ is positive for all k in the interval $(-k_0, k_0)$ and is non-positive otherwise. The implications of this as far as the issue of the ellipticity of the displacement equations of equilibrium are concerned were

observed[1] in Section 3.

We now turn to the issue of piecewise homogeneous elastostatic shocks. Suppose we are given the deformation gradient $\overset{+}{\underset{\sim}{F}}$, and hence (through (2.4), (6.5) and (6.9)) the associated value of β (say $\hat{\beta}$), and the pressure $\overset{+}{p}$ on $\overset{+}{\Pi}$. We look at the question of the existence of a corresponding elastostatic shock with spatial shock-line inclination $\hat{\phi}$ to the y_1-axis. $\hat{\phi}$ and $\hat{\beta}$ are held fixed in this discussion, and as noted previously we may assume $(\hat{\phi}, \hat{\beta})$ to be in $\mathring{\mathfrak{a}}$, with no loss of generality. We recall that a corresponding piecewise homogeneous elastostatic shock exists if and only if the function

$$h(x) = (b + cx)W'(\overset{+}{I} + 2bx + cx^2) - bW'(\overset{+}{I}) \quad , \tag{8.3}$$

where

$$\left.\begin{array}{l} b = -\dfrac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2}}\sin 2\hat{\phi} \quad < 0 \quad , \quad c = \dfrac{1-\hat{\beta}\cos 2\hat{\phi}}{\sqrt{1-\hat{\beta}^2}} \quad > 0 \quad , \\[2ex] \overset{+}{I} = \dfrac{2}{\sqrt{1-\hat{\beta}^2}} \quad > 2 \quad , \end{array}\right\} \tag{8.4}$$

has a zero at some $x \neq 0$. Using (8.1) in (8.3), we find, for the type of materials under consideration, that

$$h(x) = \frac{\mu}{2}\exp\left(-\frac{(\overset{+}{I}-2)}{2k_0^2}\right)\left\{(b+cx)\exp\left(-\frac{(2bx+cx^2)}{2k_0^2}\right) - b\right\} \quad . \tag{8.5}$$

<u>Case (i)</u> Suppose $\hat{\phi}$ and $\hat{\beta}$ are such that

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[1] See discussion following Equation (3.33).

$$b^2 > ck_0^2 \ . \tag{8.6}$$

Then $h'(0) < 0$. One shows easily from (8.4), (8.5) and (8.6) that in this case $h(x)$ has a unique zero (in addition to the one at the origin) at $x = \varkappa$, where $\varkappa$ is a positive number and is such that $\int_0^{\varkappa} h(x)dx < 0$.

It follows that, corresponding to the homogeneous deformation associated with $\hat{\beta}$ on $\overset{+}{\Pi}$ and to the inclination $\hat{\phi}$ compatible with (8.4) and (8.6), there exists a unique piecewise homogeneous elastostatic shock with positive shock strength $\varkappa$. Furthermore, suppose this piecewise homogeneous shock is embedded in a quasi-static family of shocks. Then if at the instant when the family of shocks coincides with this given shock the shock-line $\mathcal{L}$ is translating into $\overset{+}{\Pi}$, it conforms with the dissipativity inequality.

<u>Case (ii)</u> Suppose $\hat{\phi}$ and $\hat{\beta}$ are such that

$$b^2 < ck_0^2 \ . \tag{8.7}$$

Then $h'(0) > 0$. In this case, it is easily verified by virtue of (8.4), (8.5) and (8.7) that $h(x)$ has a unique zero (in addition to the one at the origin) at $x = \varkappa$, where $\varkappa$ is a negative number such that $\int_0^{\varkappa} h(x)dx > 0$.

It follows that corresponding to the homogeneous deformation associated with $\hat{\beta}$ on $\overset{+}{\Pi}$ and the inclination $\hat{\phi}$ compatible with (8.4) and (8.7), there exists a unique piecewise homogeneous elastostatic shock with negative shock strength $\varkappa$. Furthermore, suppose this piecewise homogeneous shock is embedded in a quasi-static family of shocks. Then, if at the instant when the family of shocks coincides with this given shock the shock-line $\mathcal{L}$ is translating into $\overset{-}{\Pi}$, it

conforms with the dissipativity inequality.

<u>Case (iii)</u> Suppose $\hat{\phi}$ and $\hat{\beta}$ are such that

$$b^2 = ck_0^2 \ . \tag{8.8}$$

Then $h'(0) = 0$. In this case the only zero of $h(x)$ is at the origin, from which we conclude that if the homogeneous deformation on $\overset{+}{\Pi}$ is such that the associated value of $\hat{\beta}$ and the (proposed) inclination $\hat{\phi}$ conform with (8.4) and (8.8), then there is no corresponding piecewise homogeneous elastostatic shock.

These results are best visualized on the (ϕ, β)-plane. Using (8.4) we have that

$$b^2 - ck_0^2 = \frac{-\hat{\beta}^2\cos^2 2\hat{\phi} + k_0^2\hat{\beta}\sqrt{1-\hat{\beta}^2}\cos 2\hat{\phi} + \left(\hat{\beta}^2 - k_0^2\sqrt{1-\hat{\beta}^2}\right)}{(1-\hat{\beta}^2)} \ . \tag{8.9}$$

Let Γ be the curve in the first quandrant of the $\phi - \beta$ plane whose equation is

$$\Gamma: \ \beta^2\cos^2 2\phi - k_0^2\beta\sqrt{1-\beta^2}\cos 2\phi - \left(\beta^2 - k_0^2\sqrt{1-\beta^2}\right) = 0 \ . \tag{8.10}$$

Γ separates $\overset{\circ}{\mathfrak{a}}$ into two regions as shown in Fig. 3. Case (i) refers to points in the hatched open region shown there, while Case (ii) refers to points in the unhatched open region. Points on Γ refer to Case (iii). One finds that Γ has a minimum point at (ϕ_e, β_e) where

$$\phi_e = \frac{1}{2}\cos^{-1}\left(\frac{k_0}{\sqrt{4+k_0^2}}\right) , \quad \beta_e = \frac{k_0\sqrt{k_0^2+4}}{k_0^2+2} \ . \tag{8.11}$$

From (2.41) (3.33), (8.2) and (8.4) we find that the displacement equations of equilibrium are elliptic on $\overset{+}{\Pi}$, if and only if the deformation there is such that the associated value of β is less than β_e. Suppose that the given deformation on $\overset{+}{\Pi}$ is such that the displacement equations of equilibrium are non-elliptic there. Then $\hat{\beta} \geq \beta_e$. The spatial characteristics associated with this deformation are inclined to the λ_1-principal axis of $\overset{+}{\underset{\sim}{G}}$ at angles α, which because of (3.46), (8.1) and (8.4) are given by

$$\cos 2\alpha = \frac{k_0^2\sqrt{1-\hat{\beta}^2} \pm \left(\left\{(k_0^2+2)\sqrt{1-\hat{\beta}^2}-2\right\}\left\{(k_0^2-2)\sqrt{1-\hat{\beta}^2}-2\right\}\right)^{\frac{1}{2}}}{2\hat{\beta}} . \tag{8.12}$$

Note however, that the equation of the curve Γ, (8.10), can alternatively be written as

$$\Gamma: \cos 2\phi = \frac{k_0^2\sqrt{1-\beta^2} \pm \left(\left\{(k_0^2+2)\sqrt{1-\beta^2}-2\right\}\left\{(k_0^2-2)\sqrt{1-\beta^2}-2\right\}\right)^{\frac{1}{2}}}{2\beta} . \tag{8.13}$$

It is immediately evident from a comparison of (8.12) and (8.13) that, the abscissa of the points on Γ corresponding to $\hat{\beta} \geq \beta_e$ give the spatial characteristic inclinations corresponding to the deformation associated with $\hat{\beta}$.

We now summarize our findings for the particular class of materials at hand. Corresponding to any given homogeneous deformation on $\overset{+}{\Pi}$ we can have a piecewise homogeneous elastostatic shock (provided $\overset{+}{\underset{\sim}{F}}$ is not proper orthogonal, i.e. $\hat{\beta} \neq 0$).

If, at the given deformation, the displacement equations of equilibrium are elliptic on $\overset{+}{\Pi}$, so that $\hat{\beta} < \beta_e$, the spatial shock-line

may be inclined at any angle ϕ provided it is not parallel to the principal axes of $\overset{+}{\underset{\sim}{G}}$ (i.e. $\phi \neq 0, \frac{\pi}{2}$). One can show that for such an elasto-static shock, the displacement equations of equilibrium are non-elliptic on $\bar{\Pi}$. Furthermore the corresponding shock strength is negative and a quasi-static motion from such a configuration is compatible with the dissipativity inequality if the shock moves into $\bar{\Pi}$.

On the other hand if the displacement equations of equilibrium are non-elliptic at the given deformation on $\overset{+}{\Pi}$, so that $\hat{\beta} \geq \beta_e$, the spatial shock-line may be inclined at any angle ϕ provided it is not parallel to the principal axes of $\overset{+}{\underset{\sim}{G}}$ nor parallel to the spatial characteristic directions associated with the deformation on $\overset{+}{\Pi}$. In this case the sign of the shock strength and the admissible direction of quasi-static motion depends on the specific shock-line inclination (see Fig. 3). In particular note that the admissible direction of quasi-static shock motion, for dissipativity, is governed solely by <u>whether the spatial shock-line inclination is between or outside the inclinations of the 2 spatial characteristics</u> (in the relevant quadrant) associated with the deformation on $\overset{+}{\Pi}$. The ellipticity or non-ellipticity of the displacement equations of equilibrium on $\bar{\Pi}$ also turn out to depend on the specific shock-line inclination.

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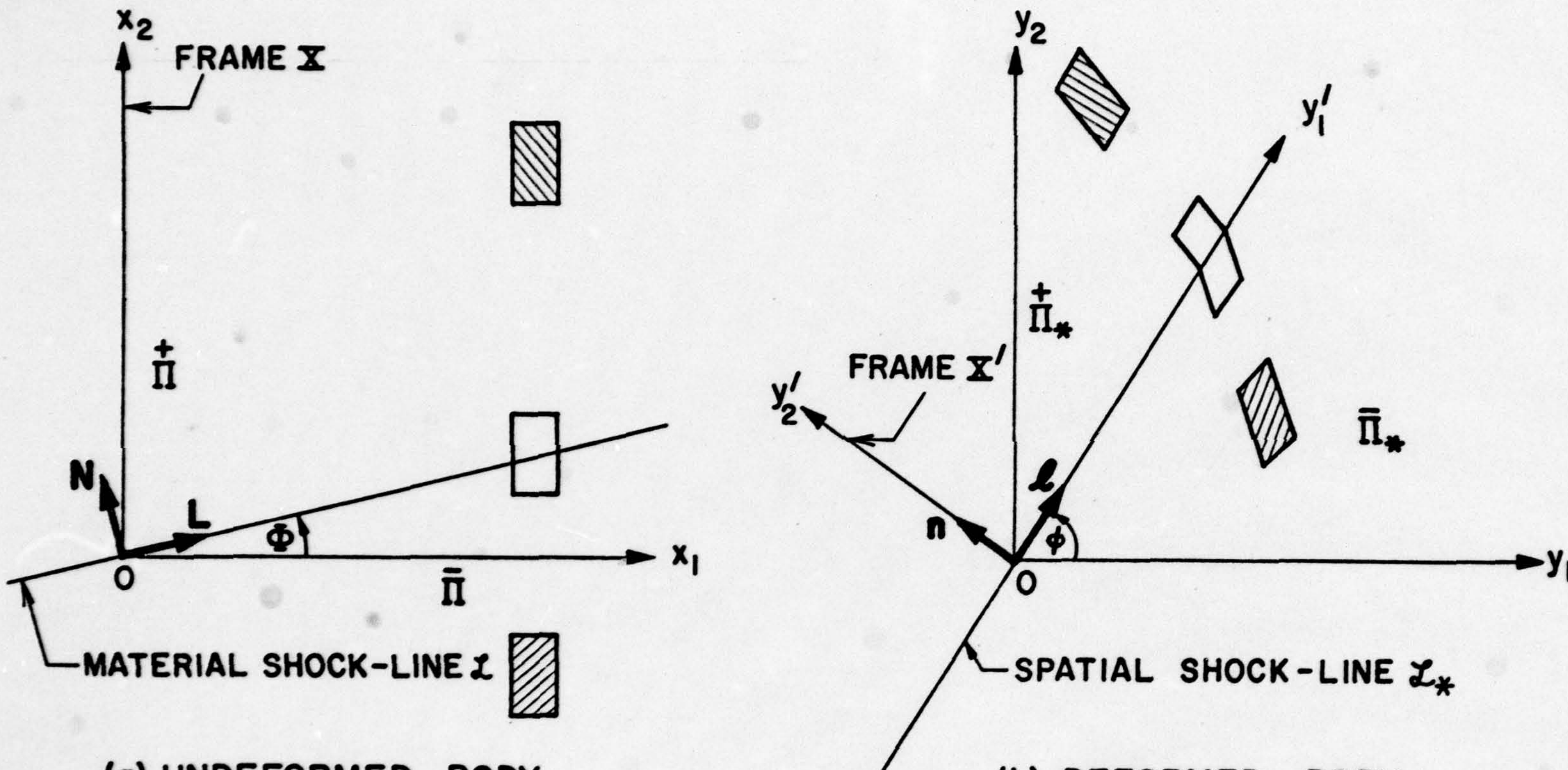

(a) UNDEFORMED BODY

(b) DEFORMED BODY

FIGURE 1. KINEMATICS OF PIECEWISE HOMOGENEOUS ELASTOSTATIC SHOCKS

$$\tau(k) = \mu k \exp(-k^2/2k_0^2)$$

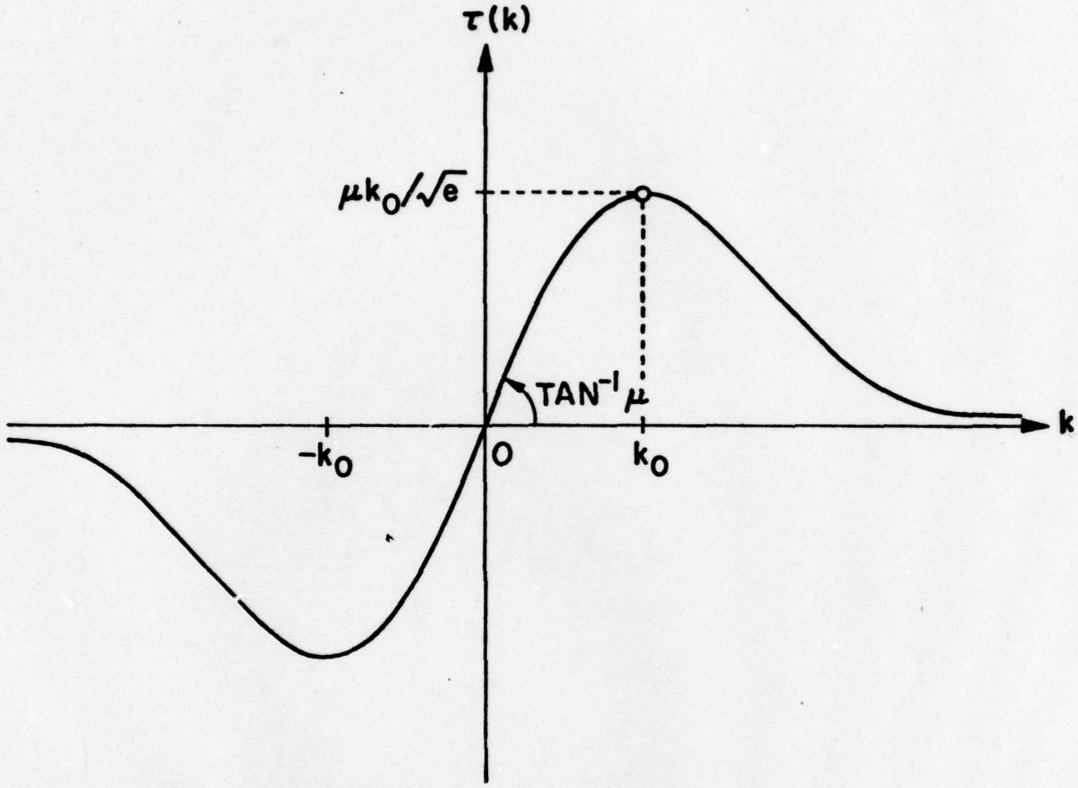

FIGURE 2. RESPONSE IN SIMPLE SHEAR.
SHEAR STRESS VS. AMOUNT OF SHEAR

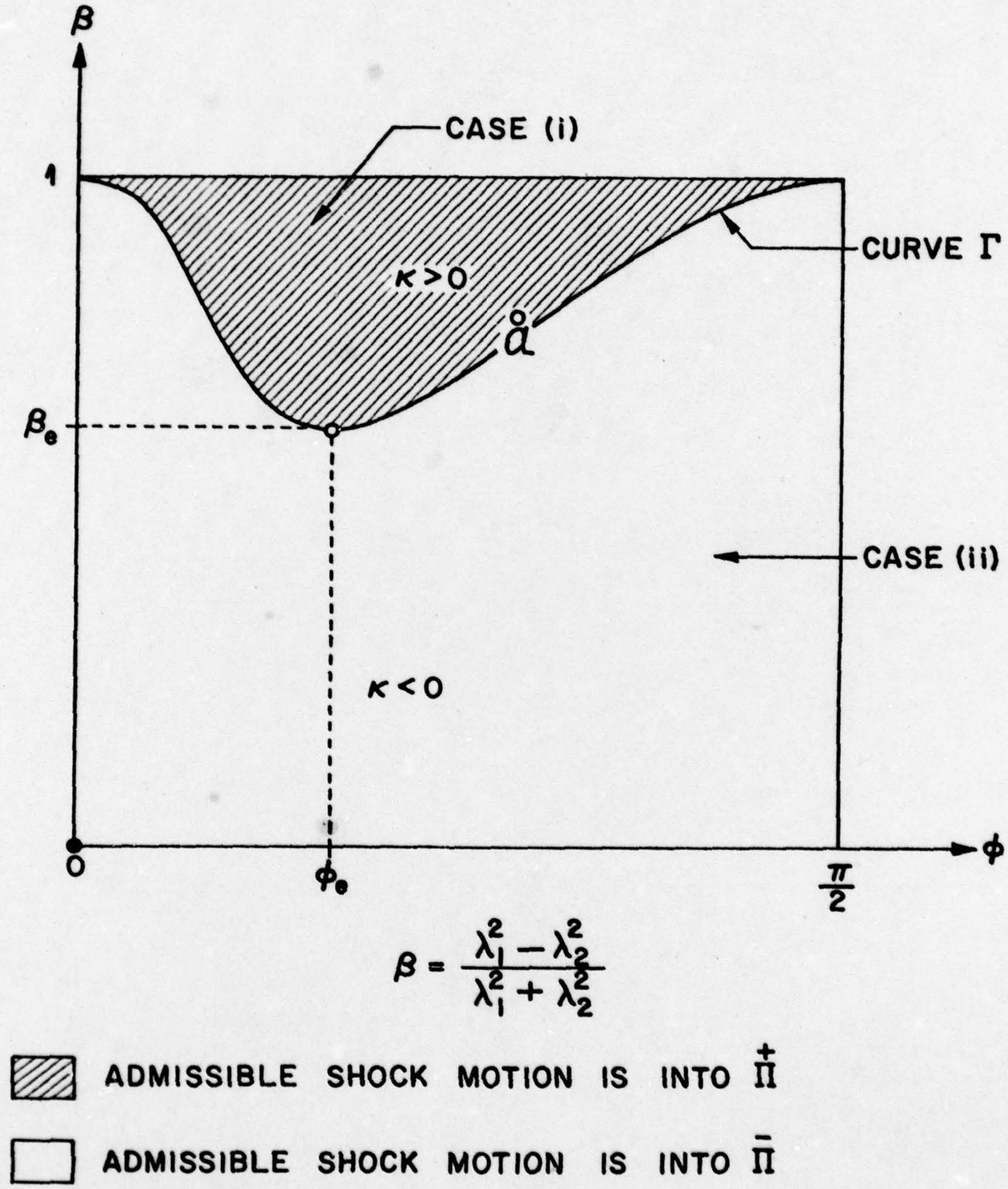

FIGURE 3. PLANE OF PARAMETERS ϕ AND β;
ADMISSIBLE REGION $\mathring{a}$

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This investigation is concerned with the possibility of the change of type of the differential equations governing finite plane elastostatics for incompressible elastic materials, and the related issue of the existence of equilibrium fields with discontinuous deformation gradients. Explicit necessary and sufficient conditions on the deformation invariants and the material for the ellipticity of the plane displacement equations of equilibrium are established.

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The issue of the existence, locally, of "elastostatic shocks" – elastostatic fields with continuous displacements and discontinuous deformation gradients – is then investigated. It is shown that an elastostatic shock exists only if the governing field equations suffer a loss of ellipticity at some deformation. Conversely, if the governing field equations have lost ellipticity at a given deformation at some point, an elastostatic shock can exist, locally, at that point. The results obtained are valid for an arbitrary homogeneous, isotropic, incompressible, elastic material.